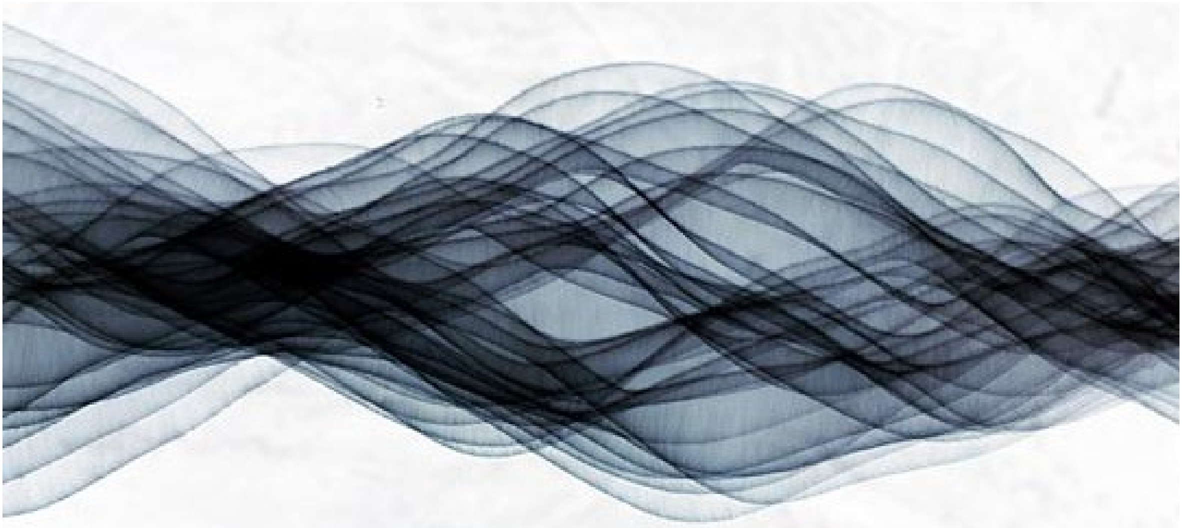

Courses in Physics

Classical Mechanics

Dynamics of Point Masses and Rigid Bodies, Vibrations and Waves, Gravity

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Part I

Classical Mechanics

Preface

Classical mechanics is the oldest and, until the discovery of electricity and magnetism and the unraveling of the structure of matter, the only discipline of physics. Since, ancient Greece powerful mathematical tools have been developed to describe the motion of bodies and stars. The successful application of these tools to new areas in recent centuries lead to the extension of physics to many other fields of research, such as electrodynamics, atomic physics, or econophysics. Nowadays, physics is better defined by the methods employed than by the subject of the studies.

Although the mathematical methods are introduced step by step along with the lecture, a first chapter 1 recapitulates the most basic notions of infinitesimal calculus, complex numbers, differential equations, and vector algebra. The subsequent chapters on the dynamics of point masses Chp. 2, on rigid bodies Chp. 3, on gravity Chp. 6, and on hydrodynamics Chp. 7 are not yet elaborated in detail. They are nevertheless included in the script because of nice exercises taken and adapted from various sources. On the other hand, the chapters on vibrations Chp. 4 and on waves Chp. 5 are ready for use.

The script contains subjects and exercises for graduation in Physics relevant to the courses of the IFSC: FCM0501 (Física I para físicos), FCM0101 (Física I para engenheiros e matemáticos), FFI0405 (Física Geral I para engenheiros e matemáticos), FCM0200 (Física Básica I para engenheiros e matemáticos), FFI0132 (Vibrações e Ondas).

The following literature is recommended for preparation and further reading:

Ph.W. Courteille, script on *Classical Mechanics: Dynamics of Point Masses and Rigid Bodies, Vibrations and Waves, Gravity* (2025)

Ph.W. Courteille, script on *Electrodynamics: Electricity, Magnetism, and Radiation* (2025)

Ph.W. Courteille, script on *Thermodynamics & Statistical Physics: applied to Gases and Solids* (2025)

Ph.W. Courteille, script on *Quantum Mechanics applied to Atoms and Light* (2025)

Ph.W. Courteille, script on *Optical Spectroscopy: A practical course* (2020)

H.J. Pain, *The Physics of Vibrations and Waves*

Zilio & Bagnato, *Apostila do Curso de Física: Mecânica, calor e ondas*

H. Moyses Nussensveig, Curso de Física Básica 2, *Fluidos, oscilações, ondas & calor*

Philippe W. Courteille, São Carlos, January 2025

Content

Chapter 1

Foundations and mathematical formalism

Before starting the physics course let us make some general comments on physics [1].

1.1 General considerations on physics

1.1.1 What is physics

1.1.1.1 Why to study physics?

The best motivation to study physics is certainly the desire to unravel the mysteries of the universe. The desire may be driven by curiosity (where are we coming from and where are we going) or by necessity, to improve our living conditions or to survive critical situations ('we gonna have to science the shit out of this' [2].)

Even if, during the studies or practicing research, curiosity is not always satisfied and many questions remain open, physicists are often confronted with questions of the very fundamental kind. Some of the questions may even have philosophical relevance (is there a master plan of the universe, is the future predetermined by the past?).

1.1.1.2 Physics and technology

Physics is the Greek word for *nature*. 'Doing physics' means setting up a *thought structure* with a proper language. The physical thought structure not only classifies observed phenomena and puts them into relation, but also allows us to predict new phenomena. The reward for this endeavor is a greater appreciation of the simplicity, beauty, and harmony of the laws of nature. But the impression to understand the world also gives us a feeling of power and supremacy, which is also alimented by the fact that physical discoveries often lead to technological applications, even though they may be sometimes unexpected.

1.1.1.3 Theory and reality

Since ancient Greece philosophers and scientists have been wondering about what is out there and how much of it human mind can grasp. How trustworthy is our

[1] See [1], Cap. 1.
[2] Matt Damon in 'The Martian'

physical thought structure? How does our image of the universe connect to reality? How much of reality can we learn through observation? After all 'it doesn't matter how beautiful your theory is, it doesn't matter how smart you are. If it doesn't agree with experiment, it's wrong' [3]. These questions have come up again with the advent quantum mechanics and its Copenhagen interpretation, which tells us what exactly we are allowed to know about nature and what features will forever be hidden in an abstract theoretical description of it. Nevertheless, 'quantum phenomena do not occur in a Hilbert space, they occur in a laboratory' [4].

1.1.1.4 The scientific method

The scientific method is characterized by an interplay between *observation and experimentation* and *abstraction and induction*. Natural phenomena are complicated and often occur as a combination or superposition of many different effects. In order to isolate particular dependencies, it is generally necessary to make an a *abstraction of inessential effects*. The isolation of pure and simple dependencies may lead to the identification of *physical laws*, which may be combined to *physical theories*.

Different physical theories describing the same phenomena may coexist. The quality of a theory is judged based on esthetics (how many laws and how many presumptions does it require? how simple is it to use?), consistency and completeness (how many phenomena does it handle and how many does it not?), and last not least, predictive power (how useful is it in practice?).

Every new theory should be better than the old ones. Nevertheless, it is important to try to determine the range of validity of the new theory. Of course, the a theory should contain the old ones in some limit, even if the basic concepts may differ radically. Indeed, old theories have their right to be (e.g. because they are more simple under certain circumstances). But it must be possible to be derived them from the newer more general theory following well-defined procedures.

1.1.1.5 Relationship between physics and mathematics

Although mathematics is the most important tool in physics, these two sciences are fundamentally different. According to *Carl Friedrich von Weizsäcker* mathematics is a *structural science* while physics is a *natural science* [3, 4]. The mathematical method is deductive. A mathematical theory is good when it is complete and intrinsically without controversy, it cannot be *correct*. It is nothing more than a tool and does not contain *reality*. For instance, axiomatic geometry remains valid, when the words 'point, straight, plane' are replaced by 'table, cup, chair'. The method of mathematics is deductive.

In contrast, physics is a *natural science*. A physical theory is good when it is *true*, which means that it describes the universe as correctly as possible. The central activity of a physicist is to *measure*. The physicist makes hypotheses that he formulates as theories, but he must verify their correctness experimentally, otherwise the theory is of no use. The physicist already has in his mind an image of how the universe works (which may or may not be correct), and this image guides his intuition in the

[3] Richard P. Feynman
[4] Asher Peres

formulation of new better theories or improved measurement techniques. According to *Karl Popper*, in contrast to pseudo-science and religious beliefs, a physical theory or model must be *falsifiable*, i.e. its correctness must be verifiable through experimentation. It is good practice to incorporate into a scientific model conjectures about its range of validity (e.g. sufficiently low velocities, small masses, or high energies). Within its range of validity the model should be universal, i.e. always and everywhere make correct predictions in the outcome of experiments. A single negative experiment is sufficient to falsify the model. A scientific model must provide recipes allowing for its falsification by experiment or at least allow for them to be formulated. An good example is Einstein's hidden variables assumption. At the beginning the debate appeared as purely metaphysical. Only Bell's proposal how to test the assumption by measurement turned it into a scientific theory, which finally was falsified experimentally. This shows that the whole art of a physicist consists in constructing meaningful experiments. The biggest task of theory is to allow questions to be asked. Measuring means: asking the *right question*. A right question is a question with possible answers, called *observables*.

1.1.1.6 Relationship between physics and other sciences

For being the most fundamental of all natural sciences, physics sets the basis for them, which is to say that any science must be compatible with physics. On the other hand, physics is very much characterized by the methods is employs, which are often reductionist in order to allow for exact calculations. Other sciences, for example chemistry and molecular biology, live from complexity, and any attempt to reduce it must lead to loss of essential information.

1.1.2 Structure of physics

1.1.2.1 Orders of magnitude, significant number of digits

The objects of physical science are measurable quantities or *observables*, which (at least in classical physics) correspond to *real* numbers. Even though complex numbers are sometimes used to represent physical quantities, only their real parts are associated with physical reality.

Many physical quantities have wide ranges of possible values (time, distances, ..), and many values are little known (for example, the number of stars in the universe. No physical quantity can be measured with absolute certainty. That is, when presenting the result of measurements, we always need to specify the estimated uncertainty or error. For example, a measurement of a mathematical constant may yield $\pi = 3.14159 \pm 0.00002$. Obviously, it is meaningless to specify a value with a higher precision as the uncertainty or to specify the uncertainty itself by more than 1 (or at most 2) digits. Hence, we always use the same number of *significant digits* for the value and its uncertainty, that is, the same number of digits behind the decimal point. If the quantity depends on several quantities, the significant digit is given by the quantity with the smallest precision.

The uncertainty of a measurement of a quantity m is a complicated issue because, in order to quantify it, in principle we would need a more precise measurement to compare with. However, what we always can do is repeat the measurement N times (if possible varying the conditions) and to calculate the mean value,

$$\bar{m} = \tfrac{1}{N}\sum_k m_k \tag{1.1}$$

of the individual measurement outcomes m_k, as well as the standard deviation,

$$\Delta m = \sqrt{\frac{1}{N-1}\sum_k (m_k - \bar{m})^2} \ . \tag{1.2}$$

The physical quantity is then specified as,

$$m = \bar{m} \pm \Delta m \ . \tag{1.3}$$

This uncertainty grasps *stochastic (or random) errors* due to an imperfect realization of the measurement limiting the *precision*. On the other hand, a conceptual flaw of the design of the measurement apparatus can lead to deviations, which often are unknown. These are called *systematic errors* and limit the *accuracy* of the apparatus, as illustrated in Fig. 1.1.

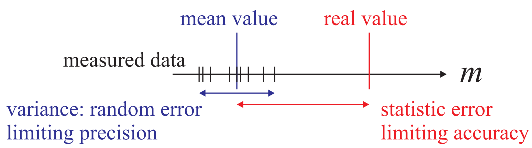


Figure 1.1: Illustration of stochastic and systematic errors in a measurement of a quantity m.

1.1.2.2 Units

Physical quantities *always* consist of a number and a unit forming an inseparable entity. For example, a mass can be specified by the assertion,

$$m = 2\,\text{kg} \ , \tag{1.4}$$

stating that a particular object I have in mind and that I will call henceforward by the name m has a mass of 2 kg. Note that every equation represents an assertion, that is either a *definition* or a *verifiable statement*, which we assume to be true if not otherwise stated. For example, the sentence

$$2\,\text{kg} \cdot 2\,\text{m/s}^2 = 4\,\text{N} \tag{1.5}$$

is a verifiable statement that we find to be correct [5]. In contrast, the statement

$$2 \ \cdot 2 = 4\,\text{N} \tag{1.6}$$

[5] A practical issue of consequently using units is that they allow us to quickly check whether a formula is false. If a formula represented by an equation exhibits different units on both sides, it must obviously be wrong.

is false and the statement

$$2 \ \cdot 2 = 4 \tag{1.7}$$

is physically meaningless.

Measuring means determining the numerical value of a physical quantity, but this is only possible with respect to a reference value. For example, distances are compared to the circumference of the Earth, times with the duration of the day. In practice, it is more convenient to choose as reference value the unitary value. For example, we will consider the distance

$$d = 1\,\mathrm{m} \ , \tag{1.8}$$

as our reference distance, which in turn is related to a (previously determined) Earth circumference by a fixed numerical factor of 1:40 000 000. In this sense, measurement always means comparison with a reference that we will call *standard*.

1.1.2.3 Linearization of functions, graphical representation

The linearization of exponential, logarithmic and polynomial functions facilitates their graphical representation and the determination of constants,

$$\begin{aligned} y &= ax + b \\ \lg y &= ax + b \qquad \Longrightarrow \quad y = 10^{ax+b} = BA^x \\ y &= a \lg x + b \\ \lg y &= a \lg x + b \quad \Longrightarrow \quad y = 10^{a \lg x + b} = Bx^a \end{aligned} \tag{1.9}$$

where we defined $B \equiv 10^b$ and $A \equiv 10^a$. See Fig. 1.2.

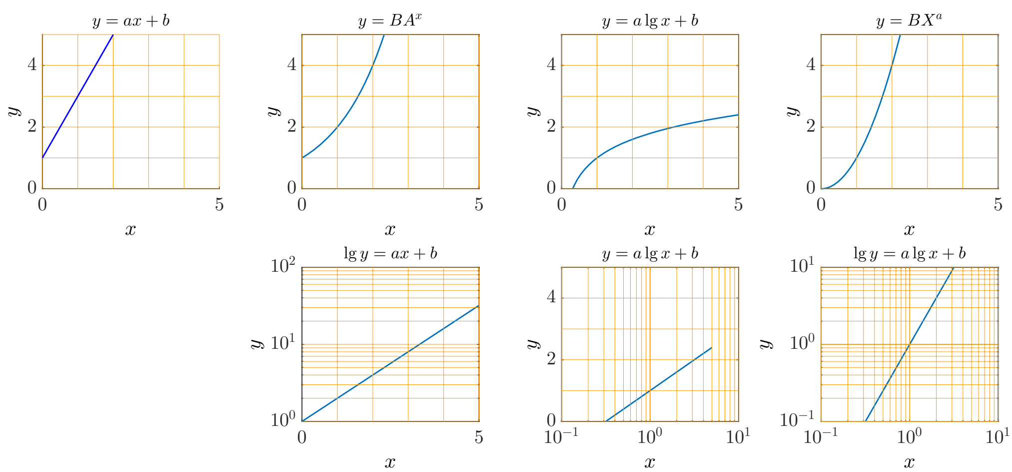


Figure 1.2: Original and linearized curved.

1.1.2.4 Length measurements

Length is the distance between two points in space and one of the 5 basic physical quantities. Its unit, the 'meter', has been introduced during the French revolution as the 40 000-th fraction of the Earth's circumference according to measurements performed at that time. The first measurement of the Earth's circumference was done by Eratosthenes 300 AC using a method explored in Exc. 1.1.3.1.

Because the original definition of the meter is not quite practical and not very precise (the Earth is not perfectly spherical and its circumference may vary over time), a physical standard meter was built out of a platinum-iridium alloy and stored at the Bureau International des Poids et Mesures (BIPM) in Paris. Its length was adjusted to fit the original definition as good as possible before it was declared the new standard.

Still, a unique physical standard is not practical, and so more abstract definitions were introduced later, such as the one using the wavelength of light emitted by a ^{86}Kr gas, according to which $1\,\mathrm{m} = 1\,650\,763.73\lambda_{\mathrm{Kr}_{86}}$. Nowadays the meter is defined via the speed of light. It is the 299 792 458-th fraction of the distance that light travels in 1 second. Of course we still need to define what a second is, which we will do later.

1.1.2.5 Coordinate systems

With the meter at hand we may subdivide a straight line in equal units marked ticks labeled by integer numbers. We may now locate a point on that line by telling how many ticks away it is from a reference point. The number of ticks is called the *coordinate* of the point. Sometimes, however, we want to localize a point on a two-dimensional plane or even three-dimensional space, which means that we need two, respectively, three coordinates.

The simplest coordinate system is constructed with three orthogonal lines crossing in one point, called *origin*. Such a coordinate system is called *Cartesian*. But other coordinate systems are possible, like *polar coordinates* in two dimensions or spherical or cylindrical coordinates in three dimensions. We will study them extensively in Sec. ??.

The reason why curvilinear coordinate systems are interesting is, that the equations describing the motion of objects confined to sub-spaces may adopt a simpler form. For example, although our Earth is embedded in 3D space, we are confined by gravity to evolve on the Earth's surface, which is a 2D curved space. Our location on Earth is thus sufficiently specified by only two numbers. The *latitude* is easy to measure, because the sun marks the location of the equator, $\lambda = 90° - \angle(\text{norte-zenit})$. On the other hand, the *longitude* is more difficult to measure and in fact needs very precise clocks.

1.1.2.6 Time measurements

Absolute space and time do not exist. According to the restricted and the general theory of relativity they are interconnected by velocity and they depend on the presence of mass. But even in a more metaphysical sense one may wonder with Gottfried Leibniz about the meaning of space with nothing inside and of time with nothing happening. Indeed, our practical approach to the measurement of time is based on the observation of recurrent phenomena that we think of being periodic, such as a day on Earth, the dripping of a water pipe, or the oscillation of a pendulum or of an atomic excitation. Assuming the time intervals separating the recurrent phenomena as being all the same, we build a ruler for time which we call clock (see Sec. ??).

Historically, the development of ever precise clocks has been motivated by navigation. Indeed, 1 minute of inaccuracy in the clock generates an uncertainty of 28 km in global positioning. And this motivation still prevails nowadays although, meanwhile, atomic clocks have uncertainties of below 10^{-16}.

Apart from periodic processes, time measurement can also be done by exponential processes, such as radioactive decay. This method is commonly used in radioactive dating. Do the Excs. 1.1.3.1 to 1.1.3.5.

1.1.2.7 The international system of units

Not every physical quantity needs to have its own standard. Often it is easier to quantify this quantity by measuring other physical quantities to which it is related in a simple way. For example, instead of measuring a *distance* by comparing it to the Earth circumference it may be easier to measure the *time* needed by a beam of light to cover this distance. In this sense, all physical quantities have been traced back by simple relationships to a small number of so-called *basic units*. Today, seven basic units are officially recognized as such by the international Conférence Générale des Poids et mesures (CGPM): the second (unit symbol 's') as a measure of time, the meter (unit symbol 'm') as a measure of distance, the kilogram (unit symbol 'kg') as a measure of mass, the Ampère (unit symbol 'A') as a measure of electrical current, the Kelvin (unit symbol 'K') as a measure of temperature, the mol (unit symbol 'mol') as a measure of molar mass, and the candela (unit symbol 'cd') as a measure of luminosity [6].

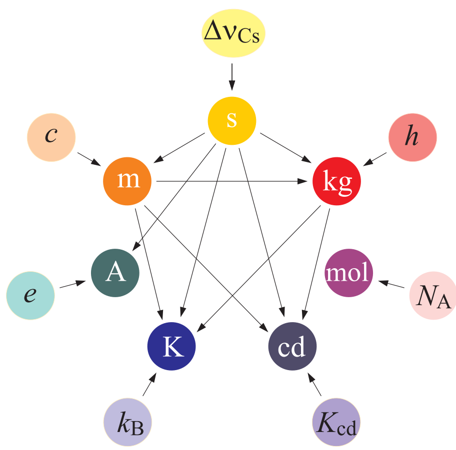


Figure 1.3: SI unit system.

[6]Note that, the molar mass and the luminosity are not fundamental in a strict sense, and only appear in the SI-system, because they are convenient for the communities of chemistry and illumination industry. After all, industrial and practical life applications are a main motivation for standardization (see Quantum Science Seminar talk by William Phillips).

By decision of the 26^e CPGM in 2018 [7], all basic units are related to the most fundamental unit, the second, for which the most precise measurement tools are available. Measurements of all other units are traced back to time measurements using simple relationships and *natural constants* [8]:

base unit	relationship	definition
time	t	$1\,\text{s} = \frac{9\,192\,631\,770}{\Delta\nu_{Cs}}$
distance	$d = ct$	$1\,\text{m} = \frac{c}{299\,792\,458}\,1\,\text{s}$
mass	$m = \frac{h\nu}{c^2}$	$1\,\text{kg} = \frac{(299\,792\,458)^2}{c^2}\frac{h}{6.626\,070\,15\times10^{-34}}\frac{1}{1\,\text{s}}$
electric current	$I = \dot{Q}$	$1\,\text{A} = \frac{e}{1.602\,176\,634\times10^{-19}}\frac{1}{1\,\text{s}}$
temperature	$T = \frac{h\nu}{k_\text{B}}$	$1\,\text{K} = \frac{c^2}{(299\,792\,458)^2}\frac{1.308\,649\times10^{-23}}{k_\text{B}}\,1\,\text{kg}$
molar mass	$T = \frac{mc^2}{k_\text{B}}$	$1\,\text{mol} = \frac{6.022\,140\,76\times10^{23}}{N_\text{A}}$
luminosity		$1\,\text{cd}$

These natural constants are the transition frequency of cesium atoms Δ_{Cs}, the velocity of light c, Planck's constant h, the elementary charge e, Boltzmann's constant k_B, and Avogadro's constant N_A. The natural constants are considered as fixed and exact. Hence, any improved measurement of the velocity of light will not result in a more precise value of c, but in an improved definition of the meter.

1.1.3 Exercises

1.1.3.1 Ex: Circumference of the Earth

In the 3rd century BC, the Greek Eratosthenes measured the circumference of the Earth by comparing the angles between the sun's rays and the vertical at noon in two different places. In Siena he measured $0°$ and in Alexandria, which is $785\,\text{km}$ north of Siena, he measured $7.2°$. What is the circumference he found?

1.1.3.2 Ex: Mean value and standard deviation

The surface area of a house plant is measured by 10 people finding different values. Determine the mean and standard deviation of the measurement. How many measurements are in the confidence interval, that is, within the standard deviation?

[7] SI-Brochure.

[8] So, in a sense, 'time' is the only remaining fundamental quantity.

person	measured value [m^2]
1	99.9
2	76
3	96.0
4	100
5	101.2
6	99.9
7	110
8	96.0
9	100.4
10	101.6

1.1.3.3 Ex: Radioactive dating with ^{40}K

A chemical analysis of a 1 g rock sample reveals the presence of $m_K = 4.21 \cdot 10^{-2}$ g potassium ($^{39}\text{K}+^{40}\text{K}$) and $m_A = 9.02 \cdot 10^{-7}$ g argon (^{40}Ar). How old is the rock? **Help:** The current relative abundance is 1 atom of ^{40}K for every 8400 atoms of ^{39}K. Only 12% of ^{40}K desintegrate into ^{40}Ar, the rest into ^{40}Ca. The decay time of ^{40}K is $t_{1/2} = 1.3 \cdot 10^9$ a.

1.1.3.4 Ex: Paramecia

A biologist analyzes the behavior of a paramecium. Make a position versus time diagram and determine the average speed.

tempo [s]	posição [cm]
0	2
10.5	2.1
12	96.0
23	100
40	101.6

1.1.3.5 Ex: Spirograph

Find the parametrization of the curves produced by a spirograph.

1.2 Infinitesimal calculus

Several physical quantities are related by derivatives or integrals.

1.2.1 Derivation

Do the Excs. 1.2.3.1 to 1.2.3.4.

1.2.2 Integration

Do the Excs. 1.2.3.5 to 1.2.3.7.

1.2.3 Exercises

1.2.3.1 Ex: Derivatives

We know that $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$. Use this notion to calculate:

$$\begin{aligned} &\text{a.} \quad (\arccos x)' \ , \\ &\text{b.} \quad (\arctan x)' \ . \end{aligned}$$

Derive by x:

$$\begin{aligned} &\text{c.} \quad y = (x^3+2)\cos x^2 \ , \\ &\text{d.} \quad y = \frac{x^2+1}{\sin\ \ln x} \ . \end{aligned}$$

Calculate the derivatives of,

$$\begin{aligned} &\text{e.} \quad f(x) = 5\ x\ e^{ax}\ \ \sin x \ , \\ &\text{f.} \quad f(t) = \frac{a^t}{\sin t} \ , \\ &\text{g.} \quad f(z) = \ln\frac{1-z^2}{1+z^2} \ . \end{aligned}$$

1.2.3.2 Ex: Curves discussion

Consider the parable $y = 2x^2 + x - 3$.
a. Using the concept of the derivative, find the position x_0 that corresponds to the extreme (maximum or minimum);
b. Substitute the value of x_0 in the parabola equation to find the value of y_0;
c. Calculate the squares to find the vertex points x_0 and y_0;
d. Find the points at which the parable crosses the axis x;
e. Sketch the parable (graphic with few details);
f. Using integration, find the area under the parable between the points 1 and 2.

1.2.3.3 Ex: Taylor series

Expand the function

$$f(x) = \frac{1}{\sqrt{1+x}}$$

in a Taylor series around the position $x = 0$. To do this, calculate the derivatives at least until third order and try to find the general law indicating how to find the series coefficients. Draw the function and the Taylor approximations.

1.2.3.4 Ex: Partial derivatives

Be $r(x_1, x_2, x_3) = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Find the first two partial derivatives and show by explicit calculation, that

$$\text{a.} \quad \frac{\partial^2 r}{\partial x_i \partial x_j} = \frac{\partial^2 r}{\partial x_j \partial x_i} \qquad \text{and} \qquad \text{b.} \quad \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \left(\frac{1}{r}\right) = 0 \; .$$

1.2.3.5 Ex: Integrals

Calculate the following indefinite integrals,

$$\text{a.} \quad \int 3x \, dx \; ,$$

$$\text{b.} \quad \int (7x^2 + 4x^3 - 2) \, dx \; ,$$

and the definite integrals,

$$\text{c.} \quad \int_0^{\pi} (3 \sin x + \cos x) \, dx \; ,$$

$$\text{d.} \quad \int_{-1}^{1} (5 + 2x^2) \, dx \; ,$$

$$\text{e.} \quad \int_0^1 e^{2x} \, dx \; ,$$

$$\text{f.} \quad \int_0^{\pi/4} \sin x \cos x \, dx \; .$$

1.2.3.6 Ex: Integrals

Find a suitable substitution to solve the following integrals,

$$\text{a.} \quad \int dt \;\; \sin(\omega t + \alpha) \qquad \text{b.} \quad \int dx \; \tfrac{2x}{\sqrt{1+x^2}}$$

$$\text{c.} \quad \int dx \; x^2 \ln x \qquad \text{d.} \quad \int_0^{\pi/3} d\phi \; \tfrac{\sin \phi}{\sqrt{1+\cos \phi}}$$

Calculate by partial integration,

$$\text{e.} \quad \int dx \; (x+2)^2 \ln x \qquad \text{f.} \quad \int dx \; xe^{-x}$$

$$\text{g.} \quad \int dx \; x^2 \ln x \qquad \text{h.} \quad \int_0^1 dx \; x\sqrt{1+x}$$

Calculate,

$$\text{i.} \quad \int \tfrac{dx}{x(\ln x)^3} \qquad \text{j.} \quad \int dx \; \tfrac{\sin 2x}{1-\cos 2x}$$

Find a serial expansion around the position $x = 0$ for,

$$\text{k.} \quad \int_0^1 dx \; e^{-x^2} \qquad \text{l.} \quad \int_0^{\pi} dx \; \tfrac{\sin x}{x}$$

1.2.3.7 Ex: Numerical integration for pedestrians

Consider the differential equation for the mass-spring system $\ddot{x} = -\omega^2 x$ with $\omega = 2\pi \cdot 1\ \mathrm{s}^{-1}$.
a. Determine the equivalent system of coupled first-order differential equations.
b. Calculate using the Euler procedure the numerical solution of this system for the initial conditions, $x(t=0) = 1$ m and $v(t=0) = 0$ for the two time intervals of $\Delta t = 0.1\,\mathrm{s}$. Compare the result with the exact solution.
c. Repeat the calculation of (b) with four time intervals of $\Delta t = 0.05\,\mathrm{s}$.
d. Use the second-order Runge-Kutta procedure with two time intervals of $\Delta t = 0.1\,\mathrm{s}$.

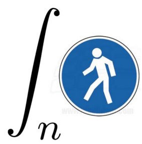


Figure 1.4: Runge-Kutta procedure.

1.3 Complex numbers

The imaginary unit is defined as,

$$\boxed{\imath \equiv \sqrt{-1}}\ . \tag{1.10}$$

From this follows a number of arithmetic rules that are studied in the following exercises.

1.3.1 Basic rules

Any complex number can be separated into a real and an imaginary part,

$$z = \mathfrak{Re}\, z + \imath \mathfrak{Im}\, z\ . \tag{1.11}$$

Both $\mathfrak{Re}\, z$ and $\mathfrak{Im}\, z$ are real numbers and may be represented graphically on a two-dimensional complex plane shown in Fig. 1.5. The complex conjugate is obtained by simply changing the sign of the imaginary part,

$$z = \mathfrak{Re}\, z - \imath \mathfrak{Im}\, z\ , \tag{1.12}$$

of more generally,

$$\bar{f}(z) = f(\bar{z})\ . \tag{1.13}$$

The absolute value of a complex number can be calculated as,

$$|z|^2 = z\bar{z} = \mathfrak{Re}^2 z + \mathfrak{Im}^2 z\ . \tag{1.14}$$

With this and defining the phase angle,

$$\varphi = \frac{\mathfrak{Im}\, z}{\mathfrak{Re}\, z} \tag{1.15}$$

We can also represent a complex number as,

$$z = |z|e^{\imath\varphi} = \cos\varphi + \imath\sin\varphi \ , \tag{1.16}$$

which is known as Euler's formula. Do the Excs. 1.3.2.1 to 1.3.2.5.

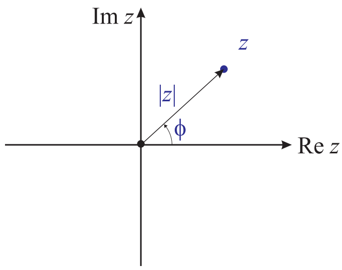


Figure 1.5: Complex plane.

1.3.2 Exercises

1.3.2.1 Ex: Complex numbers

Solve the following equations,

$$\frac{z}{1+\imath} - \frac{z}{1-\imath} = 1 + (z-\bar{z})\sin(\pi + \imath\ln 3) \quad , \quad -2\imath z = \frac{1+\bar{z}}{1+\imath} \ .$$

Calculate the absolute value, the real part and the imaginary part of,

$$\frac{2\imath - 1}{\imath - 2} \quad , \quad (1+2\imath)^3 \quad , \quad \frac{3\imath}{\imath - \sqrt{3}} \ .$$

1.3.2.2 Ex: Complex numbers

Let $z = a + \imath b$ be a complex number. Calculate the real part and the imaginary part of the following expressions: $\sin z$, $z^2 - \bar{z}^2$, $\frac{z+\imath}{\bar{z}-\imath}$.

1.3.2.3 Ex: Complex numbers

Solves the following equation by $z \in \mathbb{C}$:

$$-\bar{z} = \frac{1-z}{\imath - 1} + \ln(43)\cdot\sin\left(\frac{\pi}{\imath+1} + \frac{\pi}{\imath-1}\right) - \frac{e^{-\imath\frac{5}{4}\pi}}{\sqrt{2}} \ .$$

1.3.2.4 Ex: Complex numbers

Be $z_1 = 4 + 7\imath,\ z_2 = 3 - 9\imath$. Calculate

$$\begin{array}{ll} \text{a.} & z_1 + z_2 \quad \text{e} \quad z_1 - z_2 \\ \text{b.} & z_1 \cdot z_2 \\ \text{c.} & z_1^2 \quad \text{e} \quad z_2^3 \\ \text{d.} & |z_1| \quad \text{e} \quad |z_2| \\ \text{e.} & z_1/z_2\ . \end{array}$$

Now be $z = x + \imath y$. Calculate the real and imaginary part of the expression,

$$\frac{z}{1+2z}\ .$$

Finally, we consider two complex numbers z_1 and z_2. Show that $(z_1z_2)^* = z_1^*z_2^*$, where z^* is the conjugate complex number of z.

1.3.2.5 Ex: Complex numbers

Be

$$z_1 = 1 - \imath\sqrt{3} \quad , \quad z_2 = \frac{1}{2\sqrt{2}} + \imath\frac{\sqrt{2}}{4} \quad , \quad z_3 = -\imath\ ,$$

a. Represent $z_{1,2,3}$ in a polar form $z_{1,2,3} = r_{1,2,3}e^{\imath\phi_{1,2,3}}$.
b. Are the angles $\phi_{1,2,3}$ determined unambiguous? What values are possible?
c. Now, calculate,

$$\sqrt{z_1} \quad , \quad \sqrt[3]{z_2} \quad , \quad \sqrt[4]{z_3}\ .$$

Show how the different possible values of ϕ lead to an ambiguity of the roots.

1.4 Differential equations

1.4.1 First order differential equations

The solution is usually an exponential function. Do the Excs. 1.4.3.1 to 1.4.3.5.

1.4.2 Second order differential equations

Oscillations are processes described by second-order differential equations. We now consider the second-order linear differential equation with constant coefficients:

$$z''(x) + \alpha z'(x) + \beta z(x) = f(x)\ , \tag{1.17}$$

where $z, f : \mathcal{R} \to \mathcal{C}$, com $\alpha, \beta \in \mathcal{C}$.

The homogeneous equation $f \equiv 0$ is always solved by a linear combination of two solutions of the form $z_{1,2}(x) = e^{\lambda_{1,2}x}$ or $z_1(x) = e^{\lambda_1 x}$, $z_2(x) = xe^{\lambda_1 x}$. To show this, we consider two solutions z_1 and z_2 of the homogeneous differential equation, i.e. for

$f = 0$, then $z''_{1,2}(x) + \alpha z'_{1,2}(x) + \beta z_{1,2}(x) = 0$. Now we insert the linear combination $Az_1 + Bz_2$ into the differential equation:

$$(Az_1 + Bz_2)'' + \alpha(Az_1 + Bz_2)' + \beta(Az_1 + Bz_2) = 0 \ . \tag{1.18}$$

Arranging,

$$(Az_1'' + \alpha Az_1' + \beta Az_1) + (Bz_2'' + \alpha Bz_2' + \beta Bz_2) = 0 \ . \tag{1.19}$$

The two parentheses must disappear separately. By inserting the ansatz $e^{\lambda x}$ into the homogeneous differential equation, we get,

$$\begin{aligned}(e^{\lambda x})'' + \alpha(e^{\lambda x})' + \beta(e^{\lambda x}) &= 0 \\ \lambda^2 e^{\lambda x''} + \lambda\alpha e^{\lambda x} + \beta e^{\lambda x} &= 0 \\ \lambda^2 + \lambda\alpha + \beta &= 0 \ .\end{aligned}$$

The characteristic polynomial has two solutions,

$$\lambda_{1,2} = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - \beta} \ .$$

Depending on the values of the coefficients α and β the root can be real, zero, or imaginary. For real $\lambda_{1,2}$, the solutions $e^{\lambda x}$ describe an exponential increase or a decay. For zero roots we obtain the aperiodic limit case. For imaginary $\lambda_{1,2}$ we get a vibration.

The inhomogeneous equation, $f \neq 0$, can always be solved by the solutions mentioned above plus a particular solution $z_f(x)$. Do the Excs. 1.4.3.6 to 1.4.3.6.

1.4.3 Exercises

1.4.3.1 Ex: Exponential law

The value of a coin is divided by two each day. Derive the exponential law allowing to predict the value of the currency at any future date.

1.4.3.2 Ex: First order differential equation

A differential equation of order n is usually an equation between the first n derivatives from a function, the function itself, and its independent variables with the form $F(x, y(x), y'(x), ..., y^{(n)}(x)) = 0$. The initial conditions $(y^{(n-1)}(x_0) = y_0^{(n-1)}, ..., y(x_0) = y_0)$ must be specified to obtain an unambiguous solution. A particular case are first-order equations of the form $y'(x) + f(x)g(y(x)) = 0$ with $y(x_0) = y_0$.
a. Show that the general solution of this equation is given by $\int_{y_0}^{y} \frac{d\tilde{y}}{g(\tilde{y})} = -\int_{x_0}^{x} f(\tilde{x})d\tilde{x}$.
b. Solve the following equations:

$$\begin{aligned} y'(x) + \cos 2y \cos x = 0 \qquad &\text{with} \qquad y(0) = \pi/4 \\ \log y'(x)x + y + x^2 = 0 \qquad &\text{with} \qquad y(\infty) = 0 \ . \end{aligned}$$

Does each initial condition make sense?
c. Show that homogeneous linear first-order differential equations $y'(x)+f(x)y(x) = 0$ (with $y(x_0) = y_0$) belong to the class specified above. Find the general solution. $\left(y(x) = y_0 e^{-\int_{x_0}^{x} f(x)dx}\right)$.

1.4.3.3 Ex: Bernoulli's differential equation

Consider the equation,

$$y' = ay + by^\alpha \ ,$$

with $a, b \in \mathbb{R}$, $\alpha \neq 0, 1$ and the initial conditions $x_0 = 0$ and $y_0 = \left(\frac{1-b}{a}\right)^{1/(1-\alpha)}$. Find the solution $y(x)$.
Help: Transform the nonlinear differential equation into a first order linear differential equation by the substitution $y = z^{1/(1-\alpha)}$.

1.4.3.4 Ex: Population explosion

Develop a model for population explosion.

1.4.3.5 Ex: Population explosion

Develop a model for the Wuhan virus pandemia.

1.4.3.6 Ex: Second order differential equations

Find the general solutions $z(x)$ of the following equations:
a. $7z'' - 2\imath\sqrt{3}z' - 3z = 6$
b. $z'' - 10z' + 9z = 9x$.

1.4.3.7 Ex: Second order differential equations

Given is a second order linear differential equation with constant coefficients: $z''(x) + \alpha z'(x) + \beta z(x) = f(x)$, where z, f: $\mathbb{R} \to \mathbb{C}$, α, $\beta \in \mathbb{C}$. Show that
a. for the case $f(x) = \gamma x^n$ an ansatz $z_f(x) = \sum_{k=0}^n a_k x^k$ solves the inhomogeneous equation, where $\gamma, a_k \in \mathbb{C}$;
b. for the case, that $f(x)$ can be expanded in a Taylor series around the point x_0, the ansatz $z_f(x) = \sum_{k=0}^\infty a_k(x - x_0)^k$, $\gamma, a_k \in \mathbb{C}$, solves the inhomogeneous equation. Use linearity and the result of (a).
c. for the case $f(x) = \gamma x^n e^{\delta x}$ an ansatz $z_f(x) = e^{\delta x} \sum_{k=0}^n a_k x^k$ solves the inhomogeneous equation, where γ , δ , $a_k \in \mathbb{C}$.

1.5 Scalars, vectors and matrices

All physical quantities are represented by a conjunction of a *tensor* with numerical component and a *unit*. The tensor a by a scalar, a vector, or a matrix. Do the Excs. 1.5.4.1 to 1.5.4.8.

1.5.1 Vector algebra

Let us start with a little revision of vector algebra. A vector is a physical quantity composed of a value, a direction and a unit. For example, $\mathbf{v}$ could be the velocity of a body measured in meters per second traveling in northbound direction. Mathematically, vectors form a *vector space*, that is, an algebraic construct characterized by the existence of several operations defined by the following laws.

The addition of vectors is a *commutative* and *associative* operation, that is,

$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a} \quad \text{and} \quad (\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c}) \ . \tag{1.20}$$

The multiplication with a scalar is *commutative* and *distributive*,

$$\lambda\mathbf{a}=\mathbf{a}\lambda \quad \text{and} \quad \lambda(\mathbf{a}+\mathbf{b})=\lambda\mathbf{a}+\lambda\mathbf{b} \ . \tag{1.21}$$

The scalar product of two vectors defined by,

$$\boxed{\mathbf{a}\cdot\mathbf{b}\equiv ab\cos\theta} \ , \tag{1.22}$$

where θ is the angle between the two vectors, is *commutative* and *distributive*, but not *associative*,

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a} \quad \text{and} \quad \mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c} \quad \text{and} \quad (\mathbf{a}\cdot\mathbf{b})\mathbf{b}\neq\mathbf{a}(\mathbf{b}\cdot\mathbf{c}) \ . \tag{1.23}$$

Finally, the vector product defined by,

$$\boxed{\mathbf{a}\times\mathbf{b}\equiv ab\hat{\mathbf{e}}_n\sin\theta} \ , \tag{1.24}$$

where θ is the angle between the two vectors and $\hat{\mathbf{e}}_n$ a unit vector pointing in the direction perpendicular to $\mathbf{a}$ and $\mathbf{b}$, is *distributive*, but neither *commutative* nor *associative*,

$$\begin{aligned}\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}\neq\mathbf{b}\times\mathbf{a} \quad &\text{and} \quad \mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c} \\ &\text{and} \quad (\mathbf{a}\times\mathbf{b})\times\mathbf{c}\neq\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \ .\end{aligned} \tag{1.25}$$

Once we have chosen a basis, that is, a set of three linearly independent vectors, we can also express the vectors in terms of their components in this basis. The most common basis is the Cartesian coordinate system characterized by three fixed and orthogonal vectors, $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, and $\hat{\mathbf{e}}_z$, such that each vector can be decomposed as,

$$\mathbf{a}=a_x\hat{\mathbf{e}}_x+a_y\hat{\mathbf{e}}_y+a_z\hat{\mathbf{e}}_z \ . \tag{1.26}$$

In this representation the operations on the vector space defined above read,

$$\begin{aligned}\mathbf{a}+\mathbf{b}&=(a_x+b_x)\hat{\mathbf{e}}_x+(a_y+b_y)\hat{\mathbf{e}}_y+(a_z+b_z)\hat{\mathbf{e}}_z \\ \lambda\mathbf{a}&=\lambda a_x\hat{\mathbf{e}}_x+\lambda a_y\hat{\mathbf{e}}_y+\lambda a_z\hat{\mathbf{e}}_z \\ \mathbf{a}\cdot\mathbf{b}&=a_xb_x+a_yb_y+a_zb_z \\ \mathbf{a}\times\mathbf{b}&=\begin{pmatrix}a_yb_z-a_zb_y\\ a_zb_x-a_xb_z\\ a_xb_y-a_yb_z\end{pmatrix} \ .\end{aligned} \tag{1.27}$$

Combinations of scalar and vector products can be used to calculate other geometric quantities. An example is the *scalar triple product* defined by $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ and satisfying the following permutation rules,

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=-\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a})=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} \ . \tag{1.28}$$

Its absolute value $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$ has the meaning of the volume of the parallelepiped spanned by the three vectors. The *vector triple product* defined by $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ can be simplified,

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}) \ . \tag{1.29}$$

We verify the commutativity and the distributivity of the scalar and vector triple products in Exc. 1.5.4.9, and we train them more in Excs. 1.5.4.10 to 1.5.4.12.

1.5.2 Matrices

Matrices are two-dimensional arrays of scalars.

1.5.2.1 Properties of matrices

Important properties of matrices are the *trace*,

$$\text{Tr}\, A = \sum_{ij} A_{ij} \tag{1.30}$$

and the *determinant*,

$$\det A \equiv \ . \tag{1.31}$$

Rules for determinants,

$$\det A^\top = \det A = \frac{1}{\det A^{-1}} \quad , \quad \det(AB) = \det A \det B \tag{1.32}$$
$$\det(cA) = c^{\dim A} \det A \ .$$

1.5.2.2 Partial inversion of a matrix

It is sometimes useful to invert a matrix only partially. An example is the interconversion between scattering $\mathcal{S}$ matrices and transfer $\mathcal{T}$-matrices. Mathematically, the task consists in converting into each other two systems of linear equations [6],

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \mathcal{T} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad \Longleftrightarrow \quad \begin{pmatrix} X_1 \\ Y_2 \end{pmatrix} = \mathcal{S} \begin{pmatrix} Y_1 \\ X_2 \end{pmatrix} \tag{1.33}$$

with

$$\mathcal{T} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \tag{1.34}$$

and

$$\mathcal{S} = \text{inv}_1 T = \begin{pmatrix} M_{11}^{-1} & -M_{11}^{-1}M_{12} \\ M_{21}M_{11}^{-1} & M_{22} - M_{21}M_{11}^{-1}M_{12} \end{pmatrix} \ . \tag{1.35}$$

As an example, we would like to invert the second and third row,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \mathcal{T} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} y_1 \\ x_2 \\ x_3 \\ y_4 \end{pmatrix} = \mathcal{S} \begin{pmatrix} x_1 \\ y_2 \\ y_3 \\ x_4 \end{pmatrix} \ . \tag{1.36}$$

In order to apply the above formula, we switch the rows and columns of the $\mathcal{T}$-matrix via a unitary transform,

$$\mathcal{U} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.37}$$

so that

$$\tilde{\mathcal{T}} = \mathcal{U}^{-1}\mathcal{T}\mathcal{U} = \mathcal{U}^{-1}\begin{pmatrix} M_{22}^{11} & M_{24}^{13} \\ M_{42}^{31} & M_{44}^{33} \end{pmatrix}\mathcal{U} \equiv \begin{pmatrix} M_{33}^{22} & M_{34}^{21} \\ M_{43}^{12} & M_{44}^{11} \end{pmatrix} . \tag{1.38}$$

with

$$M_{cd}^{ab} = \begin{pmatrix} T^{ab} & T^{ad} \\ T^{cb} & T^{cd} \end{pmatrix} . \tag{1.39}$$

Now, we can apply the partial inversion formula,

$$\tilde{\mathcal{S}} = \mathrm{inv}_1\tilde{\mathcal{T}} = \begin{pmatrix} (M_{33}^{22})^{-1} & -(M_{33}^{22})^{-1}\ M_{34}^{21} \\ M_{43}^{12}\ (M_{33}^{22})^{-1} & M_{44}^{11} - M_{43}^{12}\ (M_{33}^{22})^{-1}\ M_{34}^{21} \end{pmatrix} . \tag{1.40}$$

Finally,

$$\mathcal{S} = \mathcal{U}\tilde{\tilde{\mathcal{S}}}\mathcal{U}^{-1} = \mathcal{U}\,\mathrm{inv}_1(\tilde{\mathcal{T}})\mathcal{U}^{-1} = \mathcal{U}\,\mathrm{inv}_1(\mathcal{U}^{-1}\mathcal{T}\mathcal{U})\mathcal{U}^{-1} . \tag{1.41}$$

The explicit solution can be written as,

$$\mathcal{S} = \frac{1}{\Delta_{33}^{22}}\begin{pmatrix} T^{11}\Delta_{33}^{22} + T^{12}\Delta_{31}^{23} + T^{13}\Delta_{32}^{21} & \Delta_{33}^{12} & \Delta_{22}^{13} & T^{12}\Delta_{34}^{23} + T^{13}\Delta_{32}^{24} + T^{14}\Delta_{33}^{22} \\ \Delta_{31}^{23} & T^{33} & -T^{23} & \Delta_{34}^{23} \\ \Delta_{32}^{21} & -T^{32} & T^{22} & \Delta_{32}^{24} \\ T^{41}\Delta_{33}^{22} + T^{42}\Delta_{31}^{23} + T^{43}\Delta_{32}^{21} & \Delta_{42}^{33} & \Delta_{43}^{22} & T^{42}\Delta_{34}^{23} + T^{43}\Delta_{32}^{24} + T^{44}\Delta_{33}^{22} \end{pmatrix} , \tag{1.42}$$

with the abbreviation

$$\Delta_{cd}^{ab} = \det\begin{pmatrix} T^{ab} & T^{ad} \\ T^{cb} & T^{cd} \end{pmatrix} . \tag{1.43}$$

Example 1 (*A particular case*): In particular for the case $y_2 = 0 = y_3$, the formula simplifies to,

$$\begin{pmatrix} y_1 \\ x_2 \\ x_3 \\ y_4 \end{pmatrix} = \frac{1}{\Delta_{33}^{22}}\begin{pmatrix} (T^{11}\Delta_{33}^{22} + T^{12}\Delta_{31}^{23} + T^{13}\Delta_{32}^{21})x_1 + (T^{12}\Delta_{34}^{23} + T^{13}\Delta_{32}^{24} + T^{14}\Delta_{33}^{22})x_4 \\ \Delta_{31}^{23}y_1 + \Delta_{34}^{23}y_2 \\ \Delta_{32}^{21}y_1 + \Delta_{32}^{24}y_2 \\ (T^{41}\Delta_{33}^{22} + T^{42}\Delta_{31}^{23} + T^{43}\Delta_{32}^{21})x_1 + (T^{42}\Delta_{34}^{23} + T^{43}\Delta_{32}^{24} + T^{44}\Delta_{33}^{22})x_4 \end{pmatrix} . \tag{1.44}$$

The solution for x_2 and x_3 can then we used to solve the $\mathcal{T}$-matrix equation in (1.43) directly. Do the Exc. 1.5.4.13.

1.5.3 Transformation of vectors

Of course, the definition of the vector as quantity characterized by a magnitude and a direction is unequivocal, for example, the speed $\mathbf{v}$ of a car on a road. Nevertheless, the representation of the vector depends on the orientation of the Cartesian coordinate system, which is totally arbitrary. For example, a vector given by $\mathbf{r} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z$ in one system will be described by $\mathbf{r} = x'\hat{\mathbf{e}}'_x + y'\hat{\mathbf{e}}'_y + z'\hat{\mathbf{e}}'_z$ in another system.

The behavior of vectors under transformations of coordinate systems is a very important characteristic of physical quantities and of theories governing their dynamics. For example, while classical mechanics is defined by the Galilei transform, relativistic mechanics is defined by the Lorentz transform, and we will see later that electrodynamics is incompatible with the Galilei transform. In Excs. 1.5.4.14 to 1.5.4.21 we practice the calculus with rotation matrices [9].

There are basically two things that can be done with vectors in space: *translations* and *rotations*. Both will be discussed in the following.

1.5.3.1 Translations

Translations are simply performed by adding a vector to all position vectors,

$$\mathbf{r}' = \mathcal{T}_{\text{tr}}\mathbf{r} = \mathbf{r} + \mathbf{a} \ . \tag{1.45}$$

1.5.3.2 Rotations

If the two systems are simply rotated with respect to each other, we have [10],

$$\mathbf{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \mathcal{T}_{\text{rt}}\mathbf{r} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathcal{R}\mathbf{r} \ , \tag{1.46}$$

or [11],

$$\boxed{r'_k = \sum_l R_{kl} r_l} \ . \tag{1.47}$$

Example 2 (*Rotation about the z-axis*): For example, for a rotation around the z-axis by an angle of ϕ we get,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ .$$

Rotation matrices $\mathcal{R}$ must satisfy the following requirements:

- The transformation preserves the lengths and orientations of vectors and the angles between vectors. That is, the scalar product satisfies $\mathcal{R}\mathbf{r}_2 \cdot \mathcal{R}\mathbf{r}_2 = \mathbf{r}_1 \cdot \mathbf{r}_2$.
- The transformation is orthogonal, $\mathcal{R}^{-1} = \mathcal{R}^\dagger$, and unitary, $\det \mathcal{R} = 1$.

[9] Note that the procedure is different from the one used in quantum mechanics, where any transformation needs to be described by unitary operations.

[10] By $\mathcal{T}$ we denote operations or prescriptions, while $\mathcal{R}$ is a rotation matrix.

[11] We note here that a tensor of two dimensions transforms like,

$$T'_{kl} = \sum_{l,k} R_{km} R_{ln} T_{mn} \ .$$

1.5.4 Exercises

1.5.4.1 Ex: Vectors

A room has the dimensions $3 \times 4 \times 5\,\mathrm{m}^3$. A fly departs from one of its corners and flies to the diametrically opposite corner. What is the absolute value of the displacement? Could its trajectory be less than this displacement? Choose a convenient coordinate system and express the displacement in vector form.

1.5.4.2 Ex: Vectors

Consider the vectors $\mathbf{x} = x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3$ and $\mathbf{y} = y_1\hat{\mathbf{e}}_1 + y_2\hat{\mathbf{e}}_2 + y_3\hat{\mathbf{e}}_3$. Calculate $|\mathbf{x}|$, $\mathbf{x} \cdot \mathbf{y}$ and $\mathbf{x} \times \mathbf{y}$.

1.5.4.3 Ex: Vectors

Can we combine two vectors with different absolute values and have a resulting vector with zero length? How about 3 vectors?

1.5.4.4 Ex: Vectors

Consider a moving body whose position vector is given (in cm) by $\mathbf{r} = 3\hat{\mathbf{e}}_x \cos\omega t + 4\hat{\mathbf{e}}_y \sin\omega t$.
a. Display in a scaled graph $\mathbf{r}$ at a given time t;
b. after a small time interval Δt show the new vector $\mathbf{r}$ on the same graph;
c. calculate the displacement $\Delta\mathbf{s} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$ suffered by the body during the interval Δt;
d. calculate $\mathbf{v} = \Delta\mathbf{s}/\Delta t$ and check its orientation for $\omega t = 0,\ \pi/2,\ \pi,\ 3\pi/2$;
e. calculate $\mathbf{r} \cdot \mathbf{v}$ and discuss the result;
f. calculate $\mathbf{r} \times \mathbf{v}$ and discuss the result.

1.5.4.5 Ex: Vectors

Show that the magnitude of the sum of two vectors $\mathbf{a}$ and $\mathbf{b}$ is always within the limits,

$$||\mathbf{a}| - |\mathbf{b}|| \leq |\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}| \ .$$

1.5.4.6 Ex: Vector product

Invent a method based on the cross product to calculate the area enclosed by a path delimited by the points $A \to B \to C$....

1.5.4.7 Ex: Matrix multiplication

Determine the products of the following matrices,

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \quad , \quad \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad , \quad \begin{pmatrix} 4 & 3 \\ 2 & 2 \\ 7 & -5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \ .$$

1.5.4.8 Ex: Matrix multiplication

Consider the matrices A and B and the column vector $\mathbf{x}$ given by,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad , \quad B = \begin{pmatrix} -1 & -5 & 8 \\ 2 & 1 & 0 \\ 1 & 4 & 6 \end{pmatrix} \quad , \quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} .$$

a. Calculate the vector $\mathbf{y} = A(B\mathbf{x})$.
b. Show that for arbitrary matrices A and B and vectors $\mathbf{x}$ holds $A(B\mathbf{x}) = (AB)\mathbf{x}$.
c. Let A be an arbitrary matrix with A_{ij} elements. The conjugate matrix of A^t is defined by $(A^t)_{ij} = A_{ji}$; that is, the column indexes i and those of the row j are exchanged. Show,

$$(AB)^t = B^t A^t .$$

1.5.4.9 Ex: Vector algebra

a. Show that scalar and vector products are distributive.
b. Find out whether the vector product is associative.

1.5.4.10 Ex: Vector algebra

a. Consider a unit cube with one corner fixed at the origin and generated by the vectors, $\mathbf{a} = (1, 0, 0)$, $\mathbf{b} = (0, 1, 0)$ and $\mathbf{c} = (0, 0, 1)$. Determine the angle between the diagonals passing through the center of the cube.
b. Consider the plane containing the points $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ giving in (a). Use the vector product to calculate the normal vector of this plane.

1.5.4.11 Ex: Vector algebra

a. Prove the rule $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ writing both sides in component form.
b. Prove $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$. Under what conditions holds $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$?

1.5.4.12 Ex: Vector algebra

a. Two vectors point from the origin to points $\mathbf{r} = (2, 8, 7)$ and $\mathbf{r}' = (4, 6, 8)$. Determine the distance between the points.

1.5.4.13 Ex: Partial inversion of a matrix

Consider the transformation,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \mathcal{T} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad \text{with} \qquad \mathcal{T} = \begin{pmatrix} t & & -r & \\ & t & & -r \\ -r & & t & \\ & -r & & t \end{pmatrix}$$

and invert is partially with respect to the second and third coordinate.

1.5.4.14 Ex: Rotation of the coordinate system

a. Show that the two-dimensional rotation matrix

$$\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

preserves the scalar product, that is, $a'_x b'_x + a'_y b'_y = a_x b_x + a_y b_y$.
b. What are the constraints for the elements R_{ij} of the three-dimensional rotation matrix to preserve the length of an arbitrary vector under transformation?

1.5.4.15 Ex: Rotation of the coordinate system

Find the matrix describing a rotation of $120°$ around the axis $\vec{\omega} = (1, 1, 1)$.

1.5.4.16 Ex: Rotation of the coordinate system

Consider the transformation that corresponds to an inversion of the components of the vector $\mathbf{r} \rightarrow -\mathbf{r}$ and find out, how the vector product and the triple scalar product transform under inversion.

1.5.4.17 Ex: Rotation matrices

Show that the scalar product $\mathbf{a} \cdot \mathbf{b}$ and the angle α between the two vectors are preserved when we rotate the two vectors by an angle θ around any axis.

1.5.4.18 Ex: Rotation matrices

Consider the matrix,

$$\mathcal{R} = \frac{1}{2} \begin{pmatrix} \sqrt{2} & \sqrt{2} & 0 \\ -1 & 1 & \sqrt{2} \\ 1 & -1 & \sqrt{2} \end{pmatrix} .$$

a. Show that $\mathcal{R}$ is a rotation matrix.
b. Determine the axis of rotation.
Help: The rotation axis $\mathbf{a}$ stays invariant under rotation: $\mathcal{R}\mathbf{a} = \mathbf{a}$. Use this condition.
c. Determine the rotation angle.
Help: Consider for this a vector that is perpendicular to $\mathbf{a}$.

1.5.4.19 Ex: Rotation matrices

A rotation by an angle ϕ around the z-axis is described by the rotation matrix $\mathcal{R}_z(\phi)$ with,

$$\mathcal{R}_z(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

a. Show by an explicit calculation that inverse matrix satisfies, $\mathcal{R}_z^{-1}(\phi) = \mathcal{R}_z(-\phi) = \mathcal{R}_z^t(\phi)$.
b. Show, $\mathcal{R}_z(\phi_1)\mathcal{R}_z(\phi_2) = \mathcal{R}_z(\phi_1 + \phi_2) = \mathcal{R}_z(\phi_2)\mathcal{R}_z(\phi_1)$.
c. Show, $[\mathcal{R}_z(\phi_1)\mathcal{R}_z(\phi_2)]^t = \mathcal{R}_z^t(\phi_2)\mathcal{R}_z^t(\phi_1)$.

1.5.4.20 Ex: Rotation matrices

a. Be given the basis $(\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z)$ in Cartesian coordinates. Determine the transformation matrices for cylindrical basis $(\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\varphi, \hat{\mathbf{e}}_z)$ and the spherical basis $(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\varphi)$ and their inverse matrices.
b. For both cases transform the force fields,

$$\mathbf{F}_1 = -\kappa \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \mathbf{F}_2 = \gamma \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} \qquad \mathbf{F}_3 = \delta \begin{pmatrix} -\frac{xz}{\sqrt{x^2+y^2}} \\ -\frac{y\ z}{\sqrt{x^2+y^2}} \\ \sqrt{x^2+y^2} \end{pmatrix} .$$

c. Show generally that for rotation matrices, the vectors that correspond to the columns of the matrices are mutually orthogonal. The same holds for rows of the matrices. Use the relationship, $A^t A = AA^t = 1$.

1.5.4.21 Ex: Rotation of the coordinate system

Here, we want to rotate a rod around several axes and determine, whether its final orientation depends on the rotation path. We know that the matrix

$$\mathcal{R}_z(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

describes the transformation of a vector under rotation of the coordinate system by an angle α around the axis z.
a. Show that corresponding rotations around the x-axis, respectively, the y-axis are given by,

$$\mathcal{R}_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \quad , \quad \mathcal{R}_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} .$$

b. Show that a rotation of the coordinate system around the y-axis by an angle $\alpha = \pi/2$ leads to the same result as a rotation around the z-axis by the angle $\pi/2$ followed by a rotation around x by the angle $\pi/2$ followed by a rotation around z by the angle $3\pi/2$.

1.6 Further reading

H.M. Nussenzveig, Edgar Blucher (2013), *Curso de Física Básica: Mecânica - vol 1* [ISBN]

C.F. von Weizsäcker, München (1971), *Einheit der Natur* [ISBN]

C.F. von Weizsäcker, München (2006), *Aufbau der Physik* [ISBN]

Chapter 2

Dynamics of point masses

2.1 Motion of point masses

2.1.1 One-dimensional motion

Among the various movements that we will study, the one-dimensional motion is the simplest, because all vector quantities that describe the motion are parallel. Since the motion occurs in only one dimension, only one coordinate is required to specify the position of a body at each instant of time [1].

2.1.1.1 Velocity

The rate of change of the spatial coordinates of a body is called *velocity*. It can be characterized by specifying the body's position in a time table, as shown below (stroboscopic measurement) or by a graph, as shown in Fig. 2.1.

t [s]	x [m]	v [m/s]
0	0	-
1	0.4	0.4
2	0.6	0.1
3	0.7	0.06
4	0.8	0.07

The velocity averaged over a time interval $[t_1, t_2]$,

$$\bar{v} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} \ , \tag{2.1}$$

is the angular coefficient of the slope connecting the points $(x(t_1), t_1)$ and $(x(t_2), t_2)$ of the curve (see Fig. 2.1).

A motion is called *uniform* when the velocity is constant,

$$x(t) = x_0 + v(t - t_0) \ . \tag{2.2}$$

For non-uniform (e.g. accelerated) motion the instantaneous velocity is calculated via,

$$\boxed{v(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} \equiv \frac{dx(t)}{dt} \equiv \dot{x}(t)} \ . \tag{2.3}$$

[1] See [Cap. 2, Moyses] [Cap. 2, Zilio & Bagnato].

For example, the function $y(t) = -10\,\mathrm{m/s^2} \cdot t^2 + 5\,\mathrm{m/s} \cdot t + 2\,\mathrm{m}$ parametrizes a trajectory taken by a uniform motion [2].

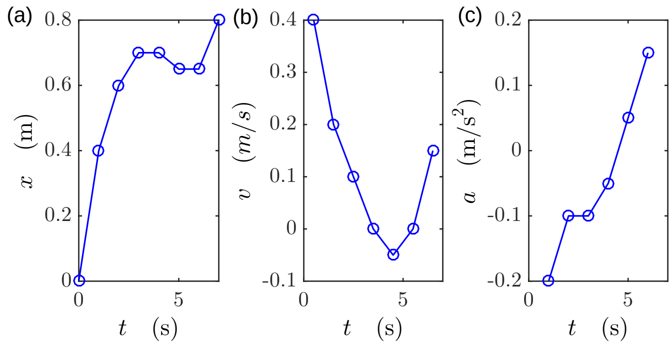


Figure 2.1: Graphical representation of the motion specified in the table. (a) Instantaneous position, (b) velocity, and (c) acceleration.

Inverse problem: Draw a graph $t \mapsto v$ and measure the area underneath:

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} v(t)dt \;. \tag{2.4}$$

2.1.1.2 Acceleration

The rate of change of the instantaneous velocity of a body is called *acceleration*. Analogous to the velocity it can be characterized by specifying the body's velocity in a time table or a graph.

Acceleration can be progressive or backward, leading to deceleration. The instantaneous acceleration is calculated via,

$$\boxed{a(t) = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t} \equiv \frac{dv(t)}{dt} \equiv \dot{v}(t)} \;. \tag{2.5}$$

Inverse problem: Draw a graph $t \mapsto a$ and measure the area underneath:

$$v(t_2) - v(t_1) = \int_{t_1}^{t_2} a(t)dt \;, \tag{2.6}$$

yielding [3],

$$x(t) = x(t_0) + \int_{t_0}^{t} v(t')dt' = x(t_0) + \int_{t_0}^{t} \left[v(t_0) + \int_{t_0}^{t'} a(t'')dt'' \right] dt' \tag{2.7}$$

$$= x(t_0) + v(t_0)(t - t_0) + \int_{t_0}^{t} \int_{t_0}^{t'} a(t'')dt''dt' \;. \tag{2.8}$$

One-dimensional motion will be studied in Excs. 2.1.4.1 to 2.1.4.11.

[2] Why do we have to write the units?
[3] Galileo.

2.1.2 Motion in two and three dimensions

2.1.2.1 Description in terms of coordinates

Under normal circumstances, space is three-dimensional, and so is the motion of a body through it. Hence, we need three coordinates to specify the position of a body at a given time, and knowing the time-dependence of the three coordinates we can parametrized the body's motion through space. Luckily, specifying a coordinate system we can reduce a bi- or (tri-)dimensional motion into two (three) one-dimensional motions.

That is to say, the physical quantities position, velocity, and force are vector quantities consistent, in fact, of three values. In this sense, they are fundamentally different from scalar quantities, such as time, temperature, or volume [4]. Using the basis,

$$\hat{\mathbf{e}}_x \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \hat{\mathbf{e}}_y \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{2.9}$$

called *Cartesian basis*, we write the vector velocity,

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \dot{x}(t)\hat{\mathbf{e}}_x + \dot{y}(t)\hat{\mathbf{e}}_y = \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} , \tag{2.10}$$

and the vector acceleration,

$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \dot{v}_x(t)\hat{\mathbf{e}}_x + \dot{v}_y(t)\hat{\mathbf{e}}_y = \begin{pmatrix} a_x(t) \\ a_y(t) \end{pmatrix} . \tag{2.11}$$

2.1.2.2 Uniformly accelerated movement by gravitation

The differential equations (2.3) or (2.5) are insufficient to determine the trajectory of a body. We additionally need to specify initial conditions. For a n-dimensional motion, we need $2n$ initial conditions.

Example 3 (*Motion due to terrestrial gravitation*): The acceleration of a body in the Earth's gravitational field is,

$$\mathbf{a}(t) = const = \begin{pmatrix} 0, -g \end{pmatrix} . \tag{2.12}$$

With arbitrary initial conditions, $\mathbf{r}(t_0) = \mathbf{r}_0$ and $\mathbf{v}(t_0) = \mathbf{v}_0$ we find,

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}(t - t_0) \tag{2.13}$$

and,

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}(t - t_0) \qquad \text{and} \qquad \mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0(t - t_0) + \tfrac{1}{2}\mathbf{a}(t - t_0)^2 . \tag{2.14}$$

Elimination time in these two equations,

$$y - y_0 = \frac{v_{0y}}{v_{0x}}(x - x_0) + \frac{a}{2}(x - x_0)^2 . \tag{2.15}$$

---

[4] Rules of how to use vectors (length, scalar product, and vector product.

Example 4 (*Motion of a projectile subject to gravity*): We identify the motion as being two-dimensional. Hence, an appropriate choice of the coordinate system allows us to eliminate one coordinate and to restrict to two coordinates that we will call $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$. We choose the initial conditions as $\mathbf{r}(t_0) = (0,0)$ and $\mathbf{v}(t_0) = v_0 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$. The equations of motion are then,

$$\mathbf{r}(t) = \begin{pmatrix} v_0 t\cos\theta \\ v_0 t \sin\theta - \frac{1}{2}gt^2 \end{pmatrix} . \tag{2.16}$$

Elimination time in the two equations (2.16),

$$y = x\tan\theta - \frac{gx^2}{2v_0^2\cos^2\theta} . \tag{2.17}$$

We my now use this equation to calculate the maximum height and the maximum range of the throw.

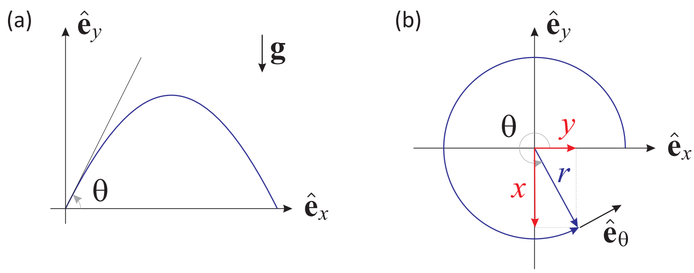


Figure 2.2: (a) Ballistic trajectory in the field of gravity. (b) Circular motion in Cartesian and polar coordinates.

Note that two movements $x(t)$ and $y(t)$ can be coupled by other physical effect, such as via the friction exerted by air.

2.1.2.3 Circular motion and polar coordinates

When we called a movement one, two, or three-dimensional, what we really meant is the number of scalar differential equations needed to describe it. Strictly speaking, however, any movement of a point mass is described by a curve in space, which is a one-dimensional object. Consequently, in the examples 3 and 4 we were able to replace the two differential equations parametrized in time $\{x(t), y(t)\}$ by a single one for $y(x)$ by eliminating time [5].

Sometimes the number of differential equations needed to describe a motion can be reduced without having to eliminate time by a proper choice of the coordinate system, which is better adapted to the symmetry of the motion. Until now we only used Cartesian coordinates (2.9). A nice example for this is the *circular motion*

[5] Note that time is automatically reintroduced in the equations once we calculate velocities by derivating coordinates.

defined by $|\mathbf{r}| = r = const$. Let us define two new coordinates oriented along the instantaneous position and the instantaneous velocity of the body,

$$\boxed{\hat{\mathbf{e}}_r(t) \equiv \frac{\mathbf{r}(t)}{r} \qquad \text{and} \qquad \hat{\mathbf{e}}_\theta(t) \equiv \frac{\mathbf{v}(t)}{v(t)}} \ . \tag{2.18}$$

We note that (for a circular motion) the new coordinates, called *polar coordinates*, are orthogonal but not fixed in space: they rotate along the circle together with the body.

The constancy of the radius $r^2 = x(t)^2 + y(t)^2$, however, implies that we may, choosing the Cartesian coordinate system such as to satisfy the initial condition $x(0) = r$, define two new variables $\{r, \theta(t)\}$ one of which is constant and the other time-dependent, such that,

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = r \begin{pmatrix} \cos\theta(t) \\ \sin\theta(t) \end{pmatrix} = r\hat{\mathbf{e}}_r(t) \ , \tag{2.19}$$

for the velocity,

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = r \begin{pmatrix} -\dot{\theta}(t)\sin\theta(t) \\ \dot{\theta}(t)\cos\theta(t) \end{pmatrix} = \dot{\theta}(t) \begin{pmatrix} -y(t) \\ x(t) \end{pmatrix} = v(t)\hat{\mathbf{e}}_\theta(t) \ , \tag{2.20}$$

and for the acceleration,

$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = r \begin{pmatrix} -\ddot{\theta}(t)\sin\theta(t) - \dot{\theta}(t)^2\cos\theta(t) \\ \ddot{\theta}(t)\cos\theta(t) - \dot{\theta}(t)^2\sin\theta(t) \end{pmatrix} = -r\dot{\theta}(t)^2\hat{\mathbf{e}}_r + r\ddot{\theta}(t)\hat{\mathbf{e}}_\theta(t) \ . \tag{2.21}$$

That is, the acceleration is composed of a tangential and a normal component,

$$\mathbf{a}(t) \cdot \hat{\mathbf{e}}_\theta(t) = r\ddot{\theta}(t) \qquad \text{and} \qquad \mathbf{a}(t) \cdot \hat{\mathbf{e}}_r = -r\dot{\theta}(t)^2 \ . \tag{2.22}$$

The circular motion is called *uniform* when $|\mathbf{v}| = v = const$. Then, $\ddot{\theta} = 0$ and $\dot{\theta} \equiv \omega = const$, and the above formulas simplify accordingly. Then $\theta = \theta_0 + \omega(t - t_0)$ and, using $\omega \equiv v/r$, we get for the arc traveled by the body,

$$s(t) = r\theta(t) = s_0 + v(t - t_0) \ . \tag{2.23}$$

Motion in one and two dimensions will be studied in Excs. 2.1.4.1 to 2.1.4.28.

2.1.3 Newton's laws

Newton formulated the following fundamental laws of classical mechanics:

i. Principle of inertia (without forces objects move with uniform velocity)

ii. $F = ma$

iii. Reaction principle (actio = reaction)

iv. Forces can be added (corollary)

Their interpretation is that: (i) In the absence of forces linear momentum is conserved; systems not subject to external forces (called *inertial systems*) are equivalent, e.g. there no way to decide an absolute speed from with in a system. (ii) Forces generate a change of linear momentum leading to an acceleration $\sum_k \mathbf{F}_k = m\dot{\mathbf{v}}$.

Newton's laws have a wide range of applications. Formulated for fixed masses, they can nevertheless be generalized to variable masses,

$$\boxed{\mathbf{F} = \dot{\mathbf{p}} = m\dot{\mathbf{v}} + \mathbf{v}\dot{m}} \ . \tag{2.24}$$

2.1.4 Exercises

2.1.4.1 Ex: Overtaking

A car travels at a constant safety distance of 40 m behind a truck (length: 25 m) at 80 km/h . To overtake, he accelerates with $a = 1.3\,\mathrm{m/s^2}$ to $v = 100\,\mathrm{km/h}$. How long is the overtaking time and the path length if the safety distance at the time of reentering the drive lane is 40 m?

2.1.4.2 Ex: Skater

A roller skater who moves at a constant speed of 10 m/s overtakes a jogger. After 5 minutes she passes a snack bar, where she takes a break for 2 minutes. Then she drives back at the same speed and meets the jogger after another 3 minutes. How fast does the jogger run?

2.1.4.3 Ex: Thieves

Two bank robbers on the run have to walk $s = 10\,\mathrm{km}$ to the rescue border, but only have one bike without luggage rack. They therefore decide to proceed as follows: Robber A drives 500 m ahead, leaves the bike and continues. Robber B goes up to the bike, sits on, drives past Robber A and continues 500 m. Then he in turn leaves the bike. In this way the robbers take turns up to the border. Assume the robbers are 6 km/h on foot and 20 km/h on bike.
a. Is this a good idea? Are the robbers really faster? If so, calculate the time saved.
b. How often do they have to change?

2.1.4.4 Ex: One-dimensional movement

A large jet plane must reach a speed of 500 km/h in order to take off, and has an acceleration of $4\,\mathrm{m/s^2}$. How long does it take for the plane to take off and how far does it have to run on the runway?

2.1.4.5 Ex: One-dimensional movement

A particle, initially at rest at the origin, moves for 10 s on a straight line, with increasing acceleration according to the law, $a = bt$, where t is time and $b = 0.5\,\mathrm{m/s^3}$. Plot the velocity v and the position x of the particle as a function of time. What is the analytical expression for $v(t)$?

2.1.4.6 Ex: One-dimensional movement

A projectile of mass m is ejected at the initial velocity v_0 vertically from a catapult.
a. Calculate the maximum height that the projectile reaches without considering the air resistance.
b. Now, we consider air resistance given by Stokes' friction law. Write down the equation of motion for this case.
c. Show that the equation of motion is solved by $v = e^{-\gamma t/m}\left(\frac{mg}{\gamma} + v_0\right) - \frac{mg}{\gamma}$.
d. Determine the time t_m when the projectile is at its maximum point from $v(t_m) = 0$.
e. What is the maximum height $s(t_m)$?

2.1.4.7 Ex: Acceleration

A possible method to measure the acceleration of gravity g is to launch a ball upwards in an evacuated tube and accurately measure the instants t_1 and t_2 of its passage (on its way up and down) across a given height z. Show that:

$$g = \frac{2z}{t_1 t_2} \ .$$

2.1.4.8 Ex: Acceleration

You want to train to be a juggler, keeping two balls in the air throwing them to a maximum height of 2 m. How often and how fast and how fast do you have to throw the balls up?

2.1.4.9 Ex: Acceleration

A racing car can be accelerated from 0 to 100 km/h in 4 s. Compare the corresponding mean acceleration with the acceleration of gravity. If the acceleration is constant, how far does the car travel until it reaches 100 km/h?

2.1.4.10 Ex: Optical cooling

Atoms of a gas (rubidium-87) are irradiated by a laser beam. Each photon transfers the moment $\Delta p = h/\lambda$, where $\lambda = 780\,\text{nm}$ is the wavelength of the light and $h = 6.626 \cdot 10^{-34}\,\text{Js}$. The mass of a rubidium atom is $m = 87 \cdot 1.66 \cdot 10^{-27}\,\text{kg}$. The scattering rate is $\Gamma = 3.8 \cdot 10^7\,\text{s}^{-1}$. What is the acceleration? Compare with the Earth's gravitational acceleration.

2.1.4.11 Ex: Free fall

A stone falls from a balloon vertically ascending at a speed of $v_0 = 20\,\text{m/s}$.
a. Calculate the speed of the stone after 10 s.
b. Calculate the distance covered by the stone in 10 s.
c. Solve the same problem with a balloon descending at the same speed.

2.1.4.12 Ex: Bodies sliding down a slope

Two children are descending with an interval of 2 s a 4 m high frictionless slide ending in a horizontal plane. What is the final distance between them?

2.1.4.13 Ex: Two-dimensional movement

A hose, with the nozzle 1.5 m above ground, is pointed upwards at an angle of 30° with the floor. The water jet hits a flower bed 15 m away.
a. At what velocity does the jet come out of the hose?
b. What is its maximum height.

2.1.4.14 Ex: One-dimensional movement

The driver of a train moving with speed v_1, sees, at a distance d ahead of him, a freight train moving in the same direction with speed v_2. He applies the brakes, transmitting the acceleration $-a$ to the train. Show that if: $d > (v_1 - v_2)^2/2a$, there will be no collision and if $d < (v_1 - v_2)^2/2a$ there will be a collision.

2.1.4.15 Ex: One-dimensional movement

Drops of water fall from a shower onto the floor 2 m below. The drops fall at regular intervals and when the first one hits the ground, the fourth is starting to fall. Determine the position of all the drops at the instant when one of them hits the floor.

2.1.4.16 Ex: One-dimensional movement

The position of a particle that moves along the x-axis depends on time according to the equation: $x = at^2 - bt^3$, x in cm, t in s.
a. At what point is x maximum?
b. What is the speed and at what moment is it zero?
c. What is the acceleration and at what time is it zero?

2.1.4.17 Ex: One-dimensional movement

A plane with speed v_0 lands on an aircraft carrier with negative acceleration $a = -A\sqrt{t}$. What is the minimum required length of the runway?

2.1.4.18 Ex: One-dimensional movement

Two bodies are located at the origin of the x-axis when $t = 0$ s. Body A has a constant speed of 2 m/s. Body B is initially at rest but subject to a constant acceleration of 1 m/s^2.
a. Represent schematically, in the same graph, the positions of bodies A and B as a function of time.
b. What is the time of collision?
c. What is the position x when the collision will occur?
d. What is the speed of body B at the time of collision?
e. At what time will the velocities of the two bodies be equal?

2.1.4.19 Ex: Cylinder rolling without sliding

Consider a cylinder of radius R rolling without sliding on a horizontal plane. The center-of-mass of the cylinder is accelerated.
a. What is the angular acceleration of the cylinder?
b. What is the rotation angle β of the cylinder as a function of time?

2.1.4.20 Ex: Relative velocity of rotating bodies

Two bodies A and B are in uniform circular movements of concentric trajectories with radii r_a and r_b and angular velocities ω_a and ω_b. Determine the relative speed between the two bodies.

2.1.4.21 Ex: Acceleration of a sliding body

Determine the acceleration of a body that slides along the thread of a screw with step h and radius R. Disregard friction and consider the body to have started from rest.

2.1.4.22 Ex: Ballistics

To launch a ball from ground over a wall of height H located at a distance S (see figure), what is the lowest initial speed with which the ball has to be launched?

2.1.4.23 Ex: Ballistics

A bullet is fired from a cannon with speed v_0. Determine the geometric region where the bullet will certainly not hit the ground.

2.1.4.24 Ex: Ballistics

An inclined plane forms an angle α with the xy-plane, as shown in the figure. A body is launched with speed v_0, forming an angle with the y-axis. Disregarding friction calculate: $x_{\max}$, $z_{\max}$, and the time it takes for the projectile to return to the y-axis.

2.1.4.25 Ex: Ballistics

A stone is launched at an initial speed of $20\,\mathrm{m/s}$. Knowing that it stayed for $2\,\mathrm{s}$ in the air, calculate:
a. the launch angle (with respect to the horizontal),
b. the maximum height reached,
c. the reach,
d. another launch angle for which the stone will have the same range. (In this case the time will be different from $2\,\mathrm{s}$).

2.1.4.26 Ex: Translation with / without friction

A body is moved with speed $v = 5\,\mathrm{m/s}$ over a horizontal plane without friction. Suddenly he encounters another plane (also without friction) inclined by an angle $\theta = 30^\circ$ and having a height of $H = 0.8\,\mathrm{m}$, as shown in Fig. 2.3.

a. At what distance d from the end of the inclined plane will the body fall?
b. What is the maximum height that the body will reach?

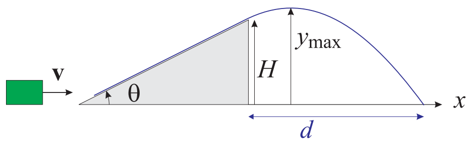


Figure 2.3:

2.1.4.27 Ex: Ballistics

A small body is launched from the origin with speed $v_0 = 100/\sqrt{3}\,\mathrm{m/s}$ under an angle $\theta = 60°$ with the horizontal. Another body is launched 1 second later, at the same speed v_0, but horizontally and from a height H, as shown in the figure. Suppose there is a collision between the two bodies and that $g = 10\,\mathrm{m/s^2}$.
a. At what time does the collision occur?
b. How high should H be for the collision to occur?
c. What are the x and y coordinates of the collision?

2.1.4.28 Ex: Ballistics

A small body is launched from the origin with speed v_0 under an angle θ with the horizontal. Another body is launched with the same speed v_0, but horizontally and from a height H. What should be the value of H such that they reach the same point on the x-axis?

2.1.4.29 Ex: Ballistics

a. Show that the movement of a projectile launched with v_0 and θ is described by the parabola: $y(x) = \frac{v_{0y}^2}{2g} - \frac{g}{2}\left(\frac{x}{v_{0x}} - \frac{v_{0y}}{g}\right)^2$, with $v_{0x} = v_0 \cos\theta$ and $v_{0y} = v_0 \sin\theta$.
b. Find the angle α that the trajectory forms with the horizontal for any x ($\tan\alpha = dy/dx$),
c. Find x_{max} corresponding to the top of the trajectory ($\tan\alpha = 0$).
d. Find the reach R by letting $\alpha = \pi - \theta$.

2.2 Kinetic and potential energy

2.2.1 Conservative potentials

Newton's axioms tell us that the reason for acceleration are forces, but they do not explain the origin of the forces. In our daily experience the forces seem to arise either from contact between bodies, i.e. collisions, or from action at a distance, as in the case of gravity or electromagnetism. In the latter case, the force exerted on a body may or may not depend on its position.

Newton realized the existence of a vertical force field called gravity, which appears to be homogeneous, that is, $F(y) = F = mgy$, where $g = 9.81\,\mathrm{m/s^2}$ is called *gravitational acceleration*.

Example 5 (*Ball rolling on a slope*): Throwing a mass into a rising slope, we experimentally notice that the height over ground Δy the mass reaches when coming to rest only depends on its initial velocity but not on the path. We also notice that this height satisfies,

$$\Delta y \propto v^2 \ . \tag{2.25}$$

Furthermore, we measure the proportionality factor to be $1/2g$.

Hence, in the homogeneous field of gravity,

$$F\Delta y = mg\Delta y = \frac{m}{2}v^2 \ . \tag{2.26}$$

We call the quantity on the right-hand side the *kinetic energy* of the mass while moving at a velocity v at ground height. The left-hand side quantity, attributed to the mass having climbed the slope to height Δy and been stopped to zero velocity, is termed *potential energy* of gravitation. Repeating the experiment for various types of slopes with various inclinations which may even vary along the slope, we verify that the final height does not depend on the trajectory of the mass, that we may parametrize as $t \to \mathbf{s}(\mathbf{t})$. That is,

$$\int_{\text{init}}^{\text{final}} \mathbf{F} \cdot d\mathbf{s} = const = V(y_{\text{final}}) - V(y_{\text{init}}) \ , \tag{2.27}$$

or for a closed path,

$$\oint \mathbf{F} \cdot d\mathbf{s} = 0 \ . \tag{2.28}$$

Only this fact allows us to attribute the potential energy solely to the mass and not to the force field. We will see later that this is only true for particular force fields (e.g. homogeneous or radial fields) and in the absence of friction. We will call force fields having this property *conservative*. The above integral equation can be cast into a differential version,

$$\boxed{\nabla \times \mathbf{F}(\mathbf{r}) = 0 \qquad \text{such that} \qquad \mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r})} \ , \tag{2.29}$$

provided, of course, that such a unique conservative potential exists.

2.2.2 Conservation of energy

Do the Excs. 2.2.4.1 to 2.2.4.20.

2.2.3 Translations and rotations of point masses, Galilei boost

The *Galilei transform* to an inertial system moving with velocity $\mathbf{u}$ is defined by,

$$\begin{aligned} t &\to t' = t \\ \mathbf{x} &\to \mathbf{x}' = \mathbf{x} + \mathbf{u}t \ . \end{aligned} \tag{2.30}$$

2.2.3.1 Galilei invariance

Consider a set of interacting particles, $m\dot{v}_i = -\nabla_{\mathbf{x}_i}\sum_j V_{ij}(|\mathbf{x}_i - \mathbf{x}_j|)$. This equation has obviously a Galilei-invariant form. In contrast, the wave equation $\nabla_r^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$ has not, because,

$$\left.\frac{\partial}{\partial t'}\right|_{r'=const} = \left.\frac{\partial}{\partial t}\right|_{r=const} - \mathbf{u}\cdot\nabla_r \ . \tag{2.31}$$

Do the Excs. 2.2.4.21 to 2.2.4.23.

2.2.4 Exercises

2.2.4.1 Ex: Potential

The following force field is given: $\mathbf{F}(\mathbf{r}) = \begin{pmatrix} -2axyz^2 + by^2 \\ -ax^2z^2 + 2bxy \\ -2ax^2yz \end{pmatrix}$. Determine the corresponding potential $V(\mathbf{r})$. Determine the work required to bring a point mass from $\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

2.2.4.2 Ex: Shot put

A shot put is able to throw a 7.257 kg ball at an initial speed of $v_0 = 15\,\mathrm{m/s}$. What is the maximum range w he can achieve with such a throw? How much further could he throw the ball (at the same initial speed) if it were half as heavy? How much further could he throw the ball if he were on the moon? **Note:** The lunar attraction is $g_M = 1.62\,\mathrm{m/s}^2$.

2.2.4.3 Ex: Shot put

In a shot put, a ball with the initial speed $v_0 = 10\,\mathrm{m/s}$ is thrown at a height of $h = 2\,\mathrm{m}$ above ground at an angle of 35°.
a. Calculate the throw distance of the ball.
b. At what speed and at what angle does the ball hit the ground?
c. Calculate the maximum throw height and the time when it is reached?
d. What condition must the throw angle fulfill for maximum throw range? At what angle is the maximum throw range reached in the special case $h = 0\,\mathrm{m}$?

2.2.4.4 Ex: Rocket in the homogeneous field of gravitation

A rocket moves in the homogeneous field of gravity vertically upwards. Reactor gas is ejected at a constant rate $\gamma = \left|\frac{dm}{dt}\right| = const$ and with the velocity v_g. Describe the recoil of the rocket exerted by the ejected gas using $\mathbf{F} = \frac{dm}{dt}\mathbf{v}_g$ or exploiting the momentum conservation law for infinitesimal changes of m (occurring with the velocity v_g) and of v_{rocket}.

2.2.4.5 Ex: Energy conservation in free fall

Consider the free fall of a mass m from the height h.
a. Write down the kinetic energy and the potential energy of the mass as functions of $z(t)$ respectively $\dot{z}(t)$.
b. Calculate $z(t)$ by integrating the energy conservation law for the initial conditions $\dot{z}(t=0)=0$. To do this, derive from $\dot{z}(t)$ an equation of the form $\dot{z}=f(z)$ and integrate this equation after separating the variables.

2.2.4.6 Ex: Ski jumping with salto

Two skiers go down a slope on different routes. Calculate the final speed and the arrival time for the two trajectories sketched on the graph by solid and dashed lines.

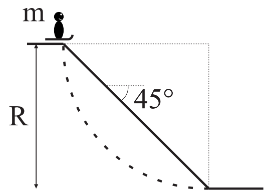


Figure 2.4: Scheme of the ramp.

2.2.4.7 Ex: Ski jumping with salto

A skier with a body weight of 75 kg descends an $L = 50$ m high ski jump with a -100% gradient. At the lower end the ski jump merges into a trough, the curvature of which is described by a circle with radius $R = 10$ m. At an inclination of 100%, the trough suddenly merges into a horizontal plane. Friction is neglected. What is the minimum weight the skis must bear before they break? How far does the skier fly over the plane D?

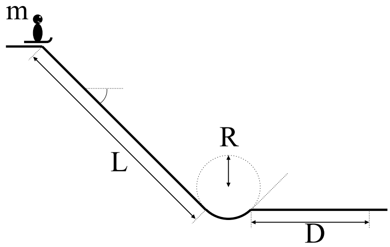


Figure 2.5: Scheme of the ramp.

2.2.4.8 Ex: Ski jumping with salto

A skier goes down a slide. The work exerted by the gravitational field on the skier is,

$$W = \int_{s_0}^{s_1} \mathbf{F} \cdot d\vec{s} = \int_0^x F_s \sqrt{1 + y''} dx \ .$$

Now we consider three different slides:
a. The slide is described by the function $y(x) = \sqrt{h^2 - x^2}$ with $dy/dx = -x/\sqrt{h^2 - x^2} = -x/y$. We also know,

$$F_s(x) = (-mg\hat{\mathbf{e}}_y) \cdot (-\hat{\mathbf{e}}_\theta) = mg\cos\theta \ .$$

Therefore, we can simplify the integral with $x = h\cos\theta$ and $y = h\sin\theta$,

$$\begin{aligned} W &= mg\cos\theta \int_0^x \frac{xdx}{\sqrt{h^2 - x^2}} = mg \int_0^x \frac{xdx}{y} = -mg \int_h^{\sqrt{h^2-x^2}} dy \\ &= mg(h - \sqrt{h^2 - x^2}) = mg(h - y) = \frac{m}{2} v^2 \ . \end{aligned}$$

b. The slide is described by the function $y(x) = (h - x)^2$ with $dy/dx = -2(h - x)$. We also know,

$$F_s(x) = -mg\sin\theta = -mg \frac{\tan\theta}{\sqrt{1 + \tan^2\theta}} = -mg \frac{y'}{\sqrt{1 + y'^2}} \ .$$

So we can simplify the integral,

$$W = -mg \int_0^x \frac{y'}{\sqrt{1 + y'^2}} \sqrt{1 + y'^2} dx = -mg \int_0^x y' dx = -mg \int_h^y dy = mg(h-y) = \frac{m}{2} v^2 \ .$$

c. The slide is described by the function $y(x) = h - \sqrt{h^2 - (x - h)^2}$. With $dy/dx = \frac{-(x-h)}{\sqrt{h^2-(x-h)^2}}$. We also know,

$$F_s(x) = -mg\sin\theta = -mg \frac{\tan\theta}{\sqrt{1 + \tan^2\theta}} = -mg \frac{y'}{\sqrt{1 + y''}} \ .$$

So we can simplify the integral,

$$W = -mg \int_0^x \frac{y'}{\sqrt{1 + y''}} \sqrt{1 + y'^2} dx = -mg \int_0^x y' dx = -mg \int_h^y dy = mg(h-y) = \frac{m}{2} v^2 \ .$$

2.2.4.9 Ex: Looping

A body starts at speed $v_0 = 0$ from point A and slides without friction on a slope (see figure). At point B the slope becomes a circle of radius R.
a. What is the speed at points B and C of the circle?
b. What is the maximum fraction R/h to prevent the body from falling at point B?
c. What is the minimum speed at point B to prevent the body from falling?

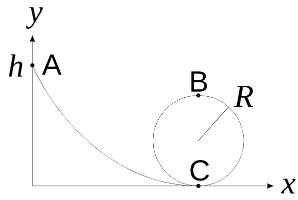


Figure 2.6: Looping.

2.2.4.10 Ex: Forces

A body of mass m moves under the influence of three forces:

$$\begin{aligned}
\mathbf{F}_1(\mathbf{r}) &= (\alpha\cos\omega(x+y), -\beta(y+y_0)^2, \gamma(z+z_0)^2) \\
\mathbf{F}_2(\mathbf{r}) &= (\alpha\sin\omega x\sin\omega y, \beta(y-y_0)^2, -\gamma(z-z_0)^2) \\
\mathbf{F}_3(\mathbf{r}) &= (\alpha\cos(\omega x+\pi)\cos\omega y, 3\beta y y_0, \gamma z_0(z_0-4z))\ .
\end{aligned}$$

The body is at $t=0$ at the point $\mathbf{r}(0)$ with velocity $\mathbf{v}(0)$. What is the absolute value of the body's velocity $|\mathbf{v}(t)|$ after a time t when it is at the point $\mathbf{r}(t)$?

2.2.4.11 Ex: Pushed inclined plane

A 45° wedge is pushed over a table with constant acceleration a. A cuboid of mass m slides on the wedge without friction. At the time $t=0$ the cuboid is at a height h and, like the cuboid, at rest.
a. Give the acceleration $\mathbf{a}(t)$ and speed $\mathbf{v}(t)$ as a function of time.
b. Give kinetic and potential energy of the cuboid as a function of time. Does energy conservation hold?

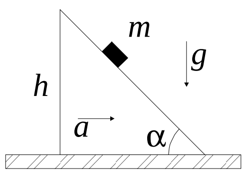


Figure 2.7: Pushed inclined plane.

2.2.4.12 Ex: Bungee jump

An organizer is offering bungee jumps from a bridge of height H. First of all, a student (body length h, mass m) wants to jump upside down from a standing position. The rope (spring constant C) is tied to him exactly in the middle of the body. Help the operator with his calculations.
a. By what length a_0 is the rope stretched when the student is hanging on the rope at rest?
b. How long is the rope to be dimensioned so that the student can just touch the

surface of the water flowing under the bridge with his head? Take into account (a) calculated pre-stretch of the rope. (Numerical values: $H = 100\,\mathrm{m}$, $h = 1.90\,\mathrm{m}$, $m = 50\,\mathrm{kg}$, $C = 50\,\mathrm{N/m}$)

2.2.4.13 Ex: Potential energy

A body is accelerated uniformly from rest until it reaches the speed v_f in time t_f. Show that the instantaneous power delivered to the body is:

$$P(t) = m v_f^2 \frac{t}{t_f^2} \ .$$

2.2.4.14 Ex: Lennard-Jones potential

Consider the Lennard-Jones potential commonly used as the interaction energy between two atoms forming a molecule:

$$U(r) = C\left[\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^{6}\right] \ .$$

a. Draw $U(r)$ as a function of r.
b. Show that the minimum energy (equilibrium position) is at r_0.
c. Find the force between atoms as a function of r.
d. What is the energy required to separate the atoms that form the molecule?

2.2.4.15 Ex: Energy conservation at a pendulum

A pendulum of mass m and length l is released from the point $\theta = 60°$ from rest. Upon reaching the vertical position $\theta = 0°$, the pendulum string encounters a nail fixed at a distance d from the ceiling. Find the minimum distance d for the mass m to rotate around the nail.

2.2.4.16 Ex: Body in a circular truss

A body of mass m moves within a vertical circular rail of radius R (see figure). When m is at the lowest position its speed is v_0.
a. What is the minimum value of v_0 such that the body goes through the entire track?
b. If v_0 were 78% of the value determined in (a), the body would go up the rail up the point P, where it will lose contact with the rail. Find the coordinate θ of this point.

2.2.4.17 Ex: Body in a potential

A body of mass M, subject to a potential $U(x) = -U_0 \cos \pi x$, is released at the origin $(x = 0)$ with speed v_0.
a. Sketch the potential in the region $-1 \leq x \leq 1$.
b. Find the force $F(x)$ acting on the body.
c. What is the maximum speed v_m that can be imparted to the body in such a way that it is confined to the region $-1 \leq x \leq 1$?

2.2.4.18 Ex: Roller coaster with looping

A mass m slides without friction along the roller coaster shown in the figure. The circular part has radius R and the mass departs from rest at point B, at the height h measured with respect to relation to the base of the tracks.
a. What is the kinetic energy of m at the point P?
b. What is the acceleration of m at point P assuming that the mass remains on the track?
c. What is the lowest value of h for m to perform the circular motion?
d. For a value of h greater than this minimum, write down the expression for the normal force exerted by the track on the mass.

2.2.4.19 Ex: Body on an inclined plane

A 2 kg body is released on an inclined plane from a point where it elongates by 4 m a spring having the constant spring constant of $k = 100\,\text{N/m}$. The spring is fixed parallel to the plane, inclined by $\theta = 30°$ (see figure).
a. Calculate the maximum compression of the spring assuming that its mass be negligible;
b. Calculate the maximum compression of the spring when the inclined plane exerts friction, the friction coefficient between it and the body being equal to 0.2);
c. In the case (b), what is the height that the body reaches on its way back upward?

2.2.4.20 Ex: Bodies climbing a slope

Two bodies propagating at the same speed v on a horizontal plane have a distance of d. After having climbed a high slope h, what will be the distance between them?

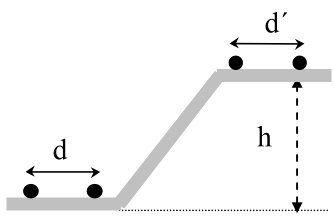


Figure 2.8: Bodies climbing a slope.

2.2.4.21 Ex: Galilei transform

A train runs at a constant speed v_0. Inside the train, from a height h_0 a ball of mass m is released.
a. Calculate the ball's trajectory as seen by an observer traveling on the train.
b. Calculate the ball's trajectory as seen by an observer outside the train.
c. Now, the train is uniformly decelerated from the moment on when the ball is released. Calculate the trajectory of the ball as seen by an observer i. traveling on the train and ii. outside the train.

2.2.4.22 Ex: Relative and center-of-mass coordinates

Consider two mass m_1 and m_2 at the positions $\mathbf{r}_1$ and $\mathbf{r}_2$.
a. Calculate the vector $\mathbf{r}$ of the center-of-mass and express the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ as a function of the vector $\mathbf{r}$ and the relative vector $\mathbf{r} \equiv= \mathbf{r}_2 - \mathbf{r}_1$.
b. Show that the vector is on the line connecting the masses m_1 and m_2. (**Help:** Parametrize the direct path from $\mathbf{r}_1$ to $\mathbf{r}_2$ and show, that $\mathbf{r}$ is a point on that path.)
c. Suppose that there are no external forces acting on the two masses and that only the gravitational force between them is present. Also suppose that each mass moves on a circle around the center-of-mass. Assume equilibrium between the gravitational attraction and the centripetal force for the two masses. Compare with the corresponding equation of motion for the relative coordinate (reduced mass).
d. Consider as an example the two-body system composed of the Earth and the Moon ($m_E = 5.974 \times 10^{24}\,\text{kg}$, $m_M = 7.35 \times 10^{22}\,\text{kg}$, $r = 384000\,\text{km}$). What is the distance d from the center-of-mass of this system from the center of the Earth?
e. What is the error in calculating the period T of a revolution of the Moon around the Earth, when the Earth's movement is neglected, that is, when the origin of the coordinates is identified with the center of the Earth?

2.2.4.23 Ex: Mobile sports field

Two athletes play with a heavy medicine ball ($m = 4\,\text{kg}$) on a moving platform without friction. They are at a distance of $L = 10\,\text{m}$. In the middle between them there is an elastic net with spring constant $D = 800\,\text{N/m}$. The left athlete throws the ball at a speed of $v_B = 8\,\text{m/s}$ parallel to the Earth's surface.
a. What is the velocity of the platform together with the athletes (total mass $M =$

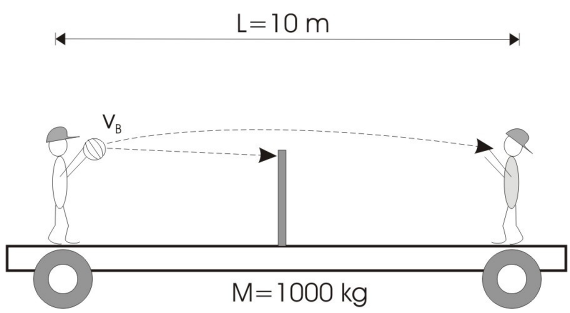


Figure 2.9: Scheme of the system.

$1000\,\text{kg}$), after the moment when the ball leaves the athlete's hand? Suppose the athletes to be firmly connected to the ground.
b. The ball reaches the athlete to the right who catches it. What is the distance covered by the platform?
c. Now, suppose that the ball is blocked by the net. What is the maximum deflection

of the net? How is the speed profile of the platform from the moment on when the ball is released?

2.3 Friction

Stokes friction $F_{\text{fr}} \propto -v$ and *Newton friction* $F_{\text{fr}} \propto -v^2$. Do the Excs. 2.3.1.1 to 2.3.1.5, 2.3.1.6 to 2.3.1.16.

2.3.1 Exercises

2.3.1.1 Ex: Free flight with friction

A body of mass m starts to fall at time $t = 0$ and point $z = 0$ with the initial velocity $\mathbf{v} = v_0\hat{\mathbf{e}}_z$ in the homogeneous terrestrial field of gravity (the z-axis is directed downwards). Its movement in the atmosphere is subject to Newton's friction force, $\mathbf{F}_R = -\alpha v \mathbf{v}$. We are considering only the movement along the z-axis.
a. What is the equation of motion for the body?
b. How should v_0 be chosen to obtain a movement with constant speed? v
c. Now, let the initial velocity v_0 be zero. Calculate the relationship between the falling distance z and the maximum reached velocity. **Help:** Use the method of separating the variables and, for that, write the equation of motion in the form $dz = f(v)dv$. Use the following result: $\int \frac{x}{a^2-x^2}dx = -0.5\ln(a^2 - x^2)$.)
d. After which falling distance does the body reach 50 % of the maximum speed?

2.3.1.2 Ex: Free flight with friction numerically

Here, we want to solve Exc. 2.3.1.1 numerically. A body of mass $m = 1\,\text{kg}$ begins to fall at time $t = 0$ at point $z = 0$ with the initial velocity $\mathbf{v} = v_0\hat{\mathbf{e}}_z$ in the homogeneous gravitational field of the Earth (the z-axis is directed downwards). When moving in the atmosphere, it is subject to Newton's friction force $\mathbf{F}_R = -\alpha v \mathbf{v}$ with $\alpha = 0.01\,\text{kg/m}$. We are considering only the movement along the z-axis.
a. What is the equation of motion for the body?
b. Calculate the trajectory of the body by numerically integrating the equation of motion with the Euler method. Use different starting velocities v_0. Choose v_0 such that, at the beginning, the movement has constant v?

2.3.1.3 Ex: Parachutist with resistance (Möllemanns Rechenaufgabe)

A skydiver exits his plane in 2000 m. Unfortunately, he forgot to put on his parachute. Suppose that air resistance can be described by Stokes friction with the friction coefficient $k = 0.05\,\text{s}^{-1}$.
a. Calculate the parachutist's maximum speed.
b. What would be the total time T of the skydiver's flight if the friction was negligible?
c. What is the distance between the skydiver and the ground at time T in case the friction is not negligible.

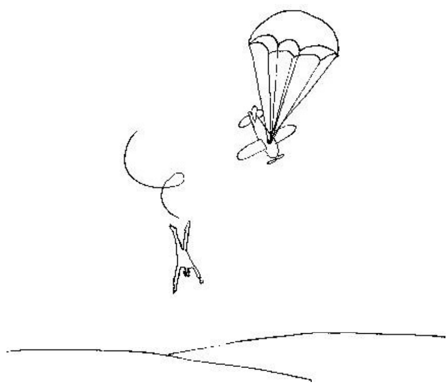


Figure 2.10: Parachutist with resistance.

2.3.1.4 Ex: Oblique throw with friction

You built a ski jumping tower, and now you have to create the associated slope. It

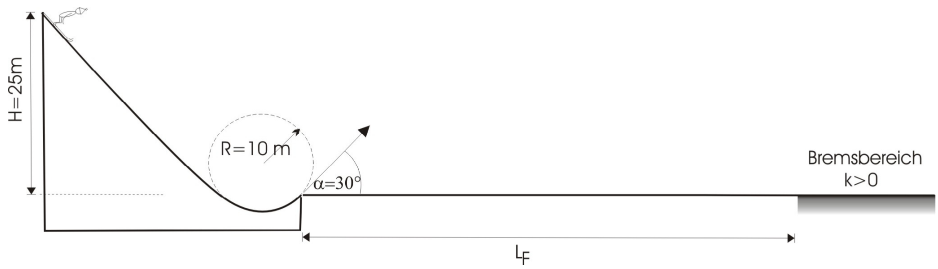


Figure 2.11: Oblique throw with friction.

consists of an inclined plane and a circular arc with radius $R = 10\,\mathrm{m}$ connected to it. The point of take-off is at ground level, and the tangent to the arc and the surface of the ground enclose an angle of $\alpha = 30°$. The ski ramp is frictionless and there is no friction by air resistance during the flight. Now, the braking area behind the desired flight path L_F, that causes a friction force $F_R = -k \cdot v$, needs to be engineered. The jumper has a mass of $m = 80\,\mathrm{kg}$.

a. At what distance L_F must the braking area begin so that the jumper arrives directly at the braking area for a height of the tower of $H = 25\,\mathrm{m}$?

b. What acceleration does the ski jumper experience in the lowest point of the tower (inside the circular path) in addition to the normal gravitational acceleration?

c. As soon as he arrives at the braking area, the frictional force acts and slows down the jumper. How large must the friction coefficient be chosen so that the ski jumper comes to a standstill in less than 6 s? A residual speed of less than 1 cm/s is defined as a standstill. To do this, set up the differential equation and solve it.

2.3.1.5 Ex: Oblique throw with friction

A soccer ball with a mass of 1 kg is shot at an initial speed v_0. In addition to the force of gravity, the Stokes force acts on the ball $\mathbf{F}_R = -km\mathbf{v}$.
a. Set up the equations of motion.
b. Solve the equations of motion for $x(t)$ and $y(t)$.
c. Show by a Taylor expansion of the solutions $x(t)$ and $y(t)$ that for $\beta \to 0$ follows the solution of the frictionless oblique throw.
d. Give an equation for the flight time T_{max} and determine the range x_{max}.
Note: Note that the general solution of the inhomogeneous differential equation consists of the general solution of the homogeneous and a particular solution of the inhomogeneous differential equation. Determine the constants from the boundary conditions.

2.3.1.6 Ex: Body sliding from a cylinder with friction

A body bound to move with friction on a cylinder of radius R is released at an angle θ_0 with zero velocity.
a. What is the maximum coefficient of static friction μ_e allowed for the body to slide?
b. Once the body starts to slide, the friction is dominated by the dynamic friction coefficient μ_d. What is the work done until the body reaches the horizontal position ($\theta = 0°$)?
c. What is the final velocity?

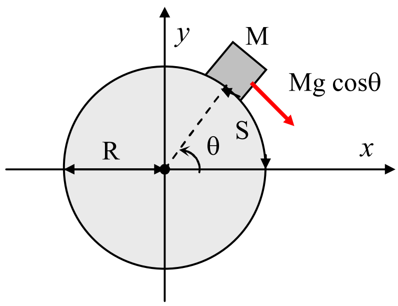


Figure 2.12: Body sliding from a cylinder with friction.

2.3.1.7 Ex: Mass sliding with friction

What is the energy expenditure for a body of mass m sliding over and decelerated by a horizontal plane with the dynamic friction coefficient μ_d.

2.3.1.8 Ex: Inclined plane with and without friction

a. Given the angle θ of an inclined plane without friction, what is the acceleration a_R such that the block of mass m shown in the figure does not slide?
b. If the inclined plane had a friction coefficient μ, what would be the maximum and minimum accelerations such that the block does not slide?

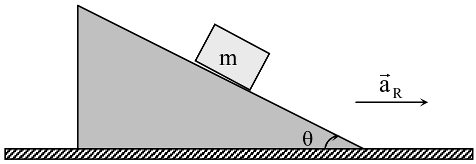


Figure 2.13: Inclined plane.

2.3.1.9 Ex: Static friction

a. The system shown in the figure is friction-free. Determine the value of the force F such that body A does not descend or rise.
b. If there were a static friction μ between the surfaces of the blocks, what would be the maximum and minimum values of forces such that body A does not fall or rise?

2.3.1.10 Ex: Inclined plane with friction

The system shown in the figure employs pulleys without mass. Find the acceleration of each block and the tension in the string.

2.3.1.11 Ex: Body subject to friction

A block of mass M rests on a table with a static friction coefficient μ_s. A force F is applied to the block so as to form an angle θ with the horizontal, as shown in the figure. Assuming that the block is always on the verge of sliding,
a. what is the angle θ that allows the applied force to be minimal, and
b. in this case, what will be the value of the force F_{min}?

2.3.1.12 Ex: Inclined plane with friction

A body of mass m is located on a triangular block of angle θ and mass M, as shown in the figure. There is no friction between the triangular block and the ground, and static friction coefficient between the two blocks is μ.
a. What is the maximum horizontal force F that can be applied to the block m such that it does not slide over the wedge?
b. What is the value of normal in this situation?

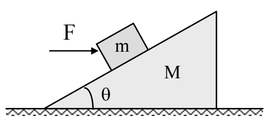


Figure 2.14: Inclined plane.

2.3.1.13 Ex: Stokes' friction

In viscous media (fluids, gases) a velocity-dependent force F_R acts opposite to the movement of a body. For laminar flows, $F_R = Cv$ with a constant C, which in the

case of a sphere of radius r has the value $C = 6\pi\eta r$, where η is the viscosity of the medium.
a. Emperor Charlemagne (year 742-814) is upset, because the Saxon Duke Widukind does let himself be subdued. Out of frustration, he threw his golden sphere, symbol of his kingdom (radius $r = 8\,\text{cm}$, density $\rho = 3.19\,\text{g/cm}^3$) into the Rhine. Establish the equation of motion for the sphere in the water and solve it with the initial values $x(t = 0) = 0\,\text{m}$, $v(t = 0) = 0\,\text{m/s}$. How fast does the golden sphere descend to the bottom? Despise buoyancy. The viscosity of water is $\eta_{H_2O} = 1.7 \cdot 10^{-3}\,\text{Ns/m}^2$.
b. If Charlemagne had launched the sphere from the Lorelei rock (height 125 m above the Rhine), how big would the difference in velocity be due to Stokes friction at the moment of impact on the Rhine's surface compared to the frictionless case? Solve the equation of motion of the sphere with the viscosity of the air $\eta_{\text{ar}} = 1.7 \cdot 10^{-5}\,\text{Ns/m}^2$.

2.3.1.14 Ex: Viscous medium

A body with initial velocity v_0 penetrates a medium that produces a viscous force given by (a) $F = -b\sqrt{v}$ and (b) $F = -cv^2$. Determine the maximum distance that the body penetrates into this medium for both cases.

2.3.1.15 Ex: Newton's friction

A sports car feels air resistance of the form $F_R = -b \cdot v^2$ as well as a rolling frictional force independent on the velocity F_{rol}. Derive an equation for the power needed to maintain a high velocity v. Assume realistic values for the constants and the sports car's power, and calculate the maximum possible resulting speed. Justify your choice of values!

2.3.1.16 Ex: Body subject to friction

Consider the system sketched in the figure, where the force F is constant and the planes have a dynamic friction coefficient μ. Calculate the total work performed by the forces acting on the system when it moves an infinitesimal distance Δx.

2.4 Many-body systems

2.4.1 Balance of forces

Do the Excs. 2.4.4.1 to 2.4.4.17.

2.4.2 Center-of-mass

We have seen earlier that the choice of the coordinate system for the description of a movement is somewhat arbitrary. This implies that the point of origin can be chosen arbitrary. For instance, we all known that the Earth is orbiting around the sun. But it would not be practical to anchor geolocalization devices to a coordinate system having the sun at its origin. Obviously, some coordinate systems are special in the sense that they simplify the mathematical description of trajectories.

One such example, is the center-of-mass coordinate system, which has the potential of greatly simplifying the description of many-body systems, e.g. planetary systems, molecules, or colliding atoms. Let us first consider two bodies of masses m_1 and m_2 located at points $\mathbf{r}_1$ and $\mathbf{r}_2$. We define the center-of mass point and the relative distance by,

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \qquad \text{and} \qquad \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \ . \tag{2.32}$$

Generalizing to many bodies,

$$\boxed{\mathbf{R} = \frac{\sum_n m_n\mathbf{r}_n}{\sum_n m_n} \qquad \text{and} \qquad \mathbf{r}_{nn'} = \mathbf{r}_n - \mathbf{r}'_n} \ . \tag{2.33}$$

The total momentum is zero in the center-of-mass coordinate system, since,

$$\mathbf{P} = \sum_n m_n\dot{\mathbf{r}}_\mathbf{n}^{\text{cm}} = \sum_n m_n\left(\dot{\mathbf{r}}_n - \frac{\sum_{n'} m_{n'}\dot{\mathbf{r}}_{n'}}{\sum_{n'} m_{n'}}\right) = 0 \ . \tag{2.34}$$

where $\mathbf{r}_n^{\text{cm}} = \mathbf{r}_n - \mathbf{R}$ are the positions of the masses with respect to the center-of-mass.

2.4.3 Collisions and conservation of linear momentum

The most universal laws of physics can be formulated as conservation laws. We have already discussed energy conservation in Sec. 2.2.2. Another conservation law holds for linear momentum. It states that for any system of many bodies not subject to external forces the total momentum cannot change,

$$\boxed{\sum_n \mathbf{p}_n = const} \ . \tag{2.35}$$

This holds if the bodies collide, even when the collisions are inelastic. Do the Excs. 2.4.4.18 to 2.4.4.32.

2.4.3.1 Binary elastic collisions and molecule formation

Binary collisions between atoms are always *elastic*. To see this, we set up the conservation laws for energy and momentum allowing for an inelastic lost of kinetic energy ΔE:

$$\boxed{\frac{p_{i1}^2}{2m} + \frac{p_{i2}^2}{2m} = \frac{p_{f1}^2}{2m} + \frac{p_{f2}^2}{2m} + \Delta E \qquad \text{and} \qquad \mathbf{p}_{i1} + \mathbf{p}_{i2} = \mathbf{p}_{f1} + \mathbf{p}_{f2}} \ . \tag{2.36}$$

Without loss of generality we may transform in to the center-of-mass system, where $\mathbf{p}'_{i1} + \mathbf{p}'_{i2} = 0 = \mathbf{p}'_{f1} + \mathbf{p}'_{f2}$. Inserting this into the law of energy conservation,

$$m\Delta E = p_{i1}'^2 - p_{f1}'^2 \ . \tag{2.37}$$

So, obviously, if no energy can be dissipated, e.g. into internal atomic excitations, the kinetic energy must remain unchanged.

Without loss of generality we may transform into the inertial system, where $\mathbf{p}''_{i2} = 0$. Inserting this into the conservation laws,

$$\frac{p''^2_{i1}}{2m} = \frac{p''^2_{f1}}{2m} + \frac{p''^2_{f2}}{2m} + \Delta E \quad \text{and} \quad \mathbf{p}''_{i1} = \mathbf{p}''_{f1} + \mathbf{p}''_{f2} \ . \tag{2.38}$$

Now substituting $\mathbf{p}_{i1}$ in the energy conservation law,

$$m\Delta E = \mathbf{p}''_{f1} \cdot \mathbf{p}''_{f2} \ . \tag{2.39}$$

So, if no energy can be dissipated, the atoms move in orthogonal directions after the collision.

Can two colliding atoms form a molecule? The laws of energy and momentum conservation require,

$$\frac{\mathbf{p}_1^2}{2m} + \frac{\mathbf{p}_2^2}{2m} = \frac{\mathbf{p}_{12}^2}{m} + \Delta E \quad \text{and} \quad \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_{12} \ . \tag{2.40}$$

In the center-of-mass inertial system, we have,

$$2\frac{\mathbf{p}''_1}{2m} = \Delta E \quad \text{and} \quad \mathbf{p}'_1 + \mathbf{p}'_2 = 0 = \mathbf{p}'_{12} \ . \tag{2.41}$$

That is, a molecule can only be formed if the collision is inelastic ($\Delta E \neq 0$), which is only possible if there is an excited internal state having exactly the energy ΔE.

2.4.4 Exercises

2.4.4.1 Ex: Mass hanging at a gallow

The mass M is hanging from a string attached to a vertical mast and attached to a boom, so that the angle becomes $\alpha = 90°$, as indicated in the figure. With $h = 1\,\text{m}$, $l = 1\,\text{m}$ and $m = 1\,\text{kg}$, calculate the tension in the wire.

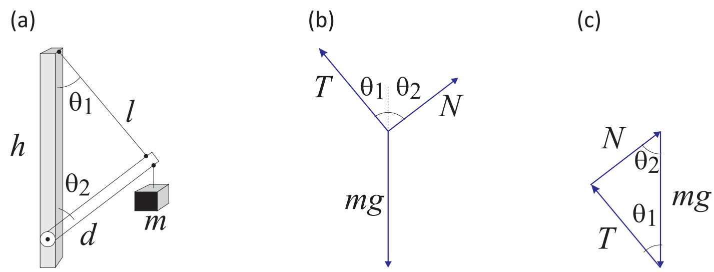


Figure 2.15: (a) Mass hanging at a gallow. (b,c) Alternative force diagrams.

2.4.4.2 Ex: Three blocks

Determine the equilibrium situation of the system sketched in the figure.

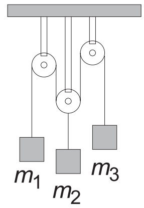


Figure 2.16: Scheme of the system.

2.4.4.3 Ex: Balance of forces on the inclined plane

One body with the mass 8 kg and a second body with the mass 10 kg connected by a wire slide without friction each one on an inclined plane. The wire runs without friction over a block, as shown in the diagram.
a. In which direction will the bodies move?
b. Calculate the acceleration of the bodies.
c. Now the two bodies are replaced by other bodies with the masses m_1 and m_2, such that there is no more acceleration, that is, the bodies are at rest. What should the ratio of the mass m_1 and m_2 be?

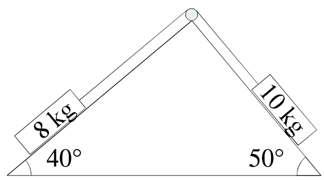


Figure 2.17: Inclined plane.

2.4.4.4 Ex: Balance of forces on the arc

Two spheres of mass m_1 and $m_2 = 2m_1$ connected by a rope of length L are located on both sides of a wall with a semicircular cross-sectional area (see figure). The diameter of the circular arc is $2R$.
a. At what horizontal distance x from the center of the wall must the spheres be in

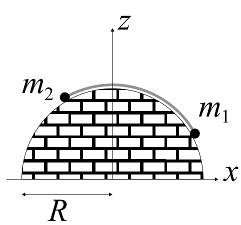


Figure 2.18: Balance of forces on the arc.

order to avoid acceleration? Assume $L = \pi R/2$.

b. Set up the equation of motion.
c. Now assume that the rope is so short that the sine expressions can be replaced by their first-order Taylor expansion. Solve the equation of motion in this approximation. Assume that at $t = 0$ the balls are the same height.

2.4.4.5 Ex: Why Paddy's Not at Work Today (Pat Cooksey)

Identify the movement described in the following song and calculate the total time of the event assuming that each floor is 3 m high, Paddy weighs 70 kg and the pile of bricks 100 kg.
Dear Sir I write this note to inform you of my plight
And at the time of writing I am not a pretty sight
My body is all black and blue, my face a deathly gray
I write this note to tell why Paddy's not at work today

While working on the fourteenth floor, some bricks I had to clear
And to throw them down from off the top seemed quite a good idea
But the gaffer wasn't very pleased, he was an awful sod
He said I had to cart them down the ladder in me hod.

Well clearing all those bricks by hand, it seemed so very slow
So I hoisted up a barrel and secured the rope below
But in my haste to do the job, I was too blind to see
That a barrel full of building bricks is heavier than me.

So when I had untied the rope, the barrel fell like lead
And clinging tightly to the rope I started up instead
I took off like a rocket and to my dismay I found
That half way up I met the bloody barrel coming down.

Well the barrel broke my shoulder as on to the ground it sped
And when I reached the top I banged the pulley with me head
I held on tight, though numb with shock from this almighty blow
And the barrel spilled out half its load fourteen floors below

Now when those building bricks fell from the barrel to the floor
I then outweighed the barrel so I started down once more
I held on tightly to the rope as I flew to the ground
And I landed on those building bricks that were scattered all around.

Now as I lay there on the deck I thought I'd passed the worst
But when the barrel reached the top, that's when the bottom burst
A shower of bricks came down on me, I knew I had no hope
In all of this confusion, I let go the bloody rope.

The barrel being heavier, it started down once more
And landed right on top of me as I lay on the floor

It broke three ribs and my left arm, and I can only say
That I hope you'll understand why Paddy's not at work today.

2.4.4.6 Ex: Superposition of forces

A climber wants to climb a mountain. He weighs with all its equipment 90 kg. He has a cable with a tensile strength of 100 kg. For safety reasons, he uses two ropes fixed at a distance of 10 m at the same height of the wall. He pulls upwards in a way to be always at an equal distance from the two fixing points. Will the ropes hold or will he fall? If so, at what point?

2.4.4.7 Ex: Coupled masses

a. Find the angle θ for the left figure so that the system remains at rest. Disregard friction.
b. Find the ratio between the masses M_1 and M_2 such that the system remains at rest in the right figure. Disregard friction.

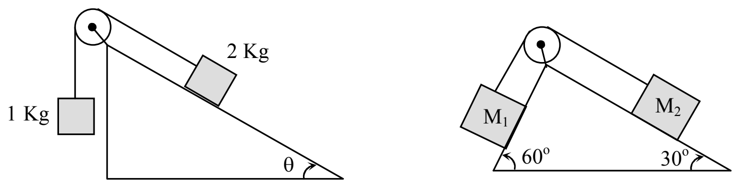


Figure 2.19: Coupled masses.

2.4.4.8 Ex: Coupled masses

a. Find the acceleration of the 2 kg body shown in the left figure.
b. Find the mass of body A such that the acceleration of body B in the right figure is zero.

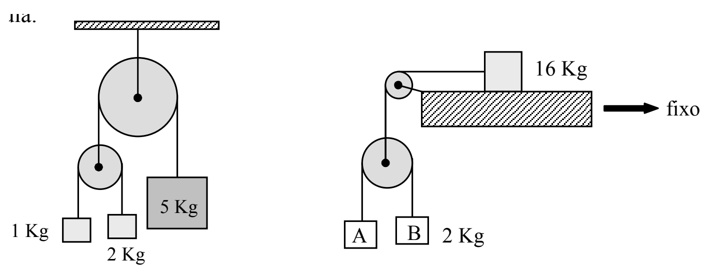


Figure 2.20: Coupled masses.

2.4.4.9 Ex: Coupled masses with friction

In the system exhibited in the figure body A slides on a surface with a friction coefficient μ. The ropes and pulleys have no mass.

a. find the accelerations of blocks A and B;
b. find the tension in the rope connected to body A.

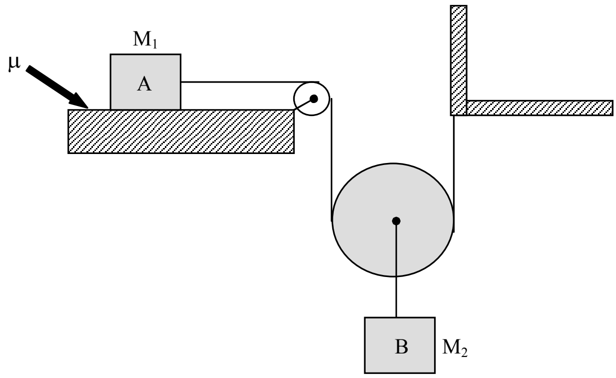


Figure 2.21: Coupled masses with friction.

2.4.4.10 Ex: Frictionless pulley

A string of length L and linear mass density λ passes through a pulley without friction. It is released from rest, with a length x pending on one side and $L - x$ on the other.
a. determine the acceleration as a function of x;
b. for which situation is the acceleration zero?

2.4.4.11 Ex: Coupled bodies

In the system shown in the figure, find:
a. the acceleration of the total system and
b. the force on the rope at point A.

2.4.4.12 Ex: Coupled bodies

N bodies connected to each other by massless strings are pulled on a ramp by means of a force F. Calculate the tension in the rope connected to the i-th body.

2.4.4.13 Ex: Coupled bodies

Consider the conical pendulum shown in the figure, where the string connecting the mass M to point O has no mass.
a. find the angle θ as a function of the velocity of mass M,
b. find the tension in the string at point O.

2.4.4.14 Ex: Coupled bodies without friction

A body of mass M is hung by an ideal string over a triangular block of angle θ, as shown in the figure. In the absence of friction between the blocks, we ask what is the maximum acceleration that can be given to the system such that the body M

remains in contact with the triangular block. In this case, what is the tension in the rope? If the system is moving at a constant speed, what is the value of the tension in the rope and of the normal?

2.4.4.15 Ex: Coupled bodies subject to friction

In the system shown in the figure, the block is in contact with the horizontal surface without friction and subject to a force F. There is a static friction μ between this block and block A in such a way that there is no relative movement between the three blocks forming the system. Calculate:
a. The angle,
b. the tension in the string, and
c. the minimum μ.

2.4.4.16 Ex: Coupled bodies with friction

A block of mass M_1 is on top of another block of mass M_2, which slides over the floor, as shown in the figure. The static friction between the two blocks is μ_e and the kinetic friction between block 2 and the ground is μ_c. a. Determine the maximum force F that can be applied to block 2 without block 1 sliding over it.
b. If the force is increased such that M_1 starts to slide, and the kinetic friction between the blocks is also μ_c, what will be the acceleration of each mass?

2.4.4.17 Ex: Coupled bodies with friction

A block of mass M is located on top of another block of the same mass, on a flat plane inclined by an angle θ, as shown in the figure. The static friction between the two blocks is μ, and between the lower block and the plane it is zero.
a. Determine the maximum force F that can be applied to the upper block without sliding over the lower block.
b. In this case, what will be the acceleration of the total system?

2.4.4.18 Ex: Firework rocket

A firework rocket is launched vertically up to a height of 50 m (along the z-axis). At the apex of the parabolic trajectory, the rocket explodes in three parts with masses $m_1 = 200\,\text{g}$, $m_2 = 300\,\text{g}$ and $m_3 = 400\,\text{g}$. The energy liberated by the explosion (200 J) is converted into kinetic energy for the three fragments. Assume that the resulting momenta are on a single horizontal line (along the x-axis). How long after the explosion and at what positions do the fragments hit the ground in the case that the larger fragment receives the same energy than the sum of the energy of the smaller fragments?
How will the results change when the masses m_1 and m_2 stay the same, but the larger fragment has the mass $m_3 = 500\,\text{g}$?

2.4.4.19 Ex: Ballistic pendulum

We measure the speed of a bullet by a ballistic pendulum. The pendulum, suspended on a wire (length $L = 2\,\text{m}$, mass $M = 4\,\text{kg}$) is moved due to the collision. Determine from the maximum horizontal displacement ($y = 10\,\text{cm}$) the velocity of the pendulum after the collision and the velocity of the bullet (mass $m = 0.1\,\text{kg}$).

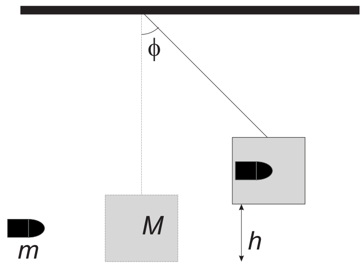


Figure 2.22: Ballistic pendulum.

2.4.4.20 Ex: Central collision in a train station

A wagon of mass m_1 collides elastically with another wagon at rest of mass m_2. What is the relation between the masses m_1 and m_2, if after the collision
a. the wagons have the same velocity in opposite directions,
b. m_2 is three times the velocity of m_1 in the same direction,
c. m_1 is repelled with a third of the initial velocity?

2.4.4.21 Ex: (In-) elastic collision

A billiard ball with the speed $\mathbf{v}_{1i} = v_0\hat{\mathbf{e}}_x$ hits a second one initially at rest. Due to the collision the first ball changes the speed to $\mathbf{v}_{1f} = \frac{2v_0}{3}(\hat{\mathbf{e}}_x \cos\theta + \hat{\mathbf{e}}_y \sin\theta)$, where $\theta = 45°$.
a. Determine the speed of the second ball, $\mathbf{v}_{2f}$, after the collision.
b. Calculate the dissipated energy and determine if shock whether elastic or inelastic.

2.4.4.22 Ex: Variable mass

A wagon of mass M initially at rest is filled with fuel representing half of its mass. The fuel is now horizontally ejected at a constant rate γ and a constant speed v_c, thus propelling the car in the opposite direction. What is the final speed of the wagon when the fuel is completely consumed?

2.4.4.23 Ex: Mass sliding down a movable wedge

A block of mass m rests on a wedge of mass M and angle θ, which is placed on a horizontal surface. Releasing the system from rest, with the body at a height h, determine the speed of the wedge when the block touches the ground. All surfaces are free from friction.

2.4.4.24 Ex: Vertical collision

A body of mass $M = 400\,\mathrm{g}$ is released from rest from a height of $h = 10\,\mathrm{m}$ with respect to the Earth's surface. Simultaneously, a bullet of mass $m = 100\,\mathrm{g}$ is fired vertically from the surface with the speed $v_0 = 10\,\mathrm{m/s}$. Knowing that somewhere in the trajectory the masses collide and merge, we ask how long it takes for the masses to fall from the moment when M is released.

2.4.4.25 Ex: Variable mass

An open freight car weighs 10 tons and is sliding on a frictionless track with a speed of $60\,\mathrm{cm/s}$. Heavy rain suddenly starts and the drops fall vertically with a velocity v_0 with respect to the ground. How fast is the wagon after collecting $500\,\mathrm{kg}$ of water?

2.4.4.26 Ex: Variable mass

An hourglass is placed on a scale plate. Initially ($t = 0$), all the sand is at rest in the upper container of the hourglass. The sand falls into the lower container at a rate $\lambda = -dm/dt$. Prepare a graph of the reading on the scale for $t \geq 0$.

2.4.4.27 Ex: Variable mass

A raindrop of initial mass M_0 starts to fall from rest. Assuming that the drop gains mass as it passes through the clouds at a rate proportional to the product of mass by velocity ($dm/dt = Kmv$), calculate the velocity $v(t)$. Neglect air resistance.

2.4.4.28 Ex: Variable mass

A toy rocket has a carcass that weighs $100\,\mathrm{g}$ and an initial amount of fuel of $400\,\mathrm{g}$. The velocity relative to the rocket with which the fuel comes out is $100\,\mathrm{m/s}$ and the rate at which it is burned is $100\,\mathrm{m/s}$. Assuming it takes off from the Earth's surface ($g = 10\,\mathrm{m/s^2}$ constant) with zero initial velocity, what maximum velocity will it reach?

2.4.4.29 Ex: Variable mass

Calculate the power required to lift a rope vertically, initially completely wound on the ground, at a constant speed v_0. The linear mass density of the rope is λ.

2.4.4.30 Ex: Collision of two masses

Two trolleys with masses m_1 and m_2 and speeds v_1 and v_2 collide elastically (energy is conserved). Knowing that the momentum of the system is preserved during the collision, calculate the speeds of the cars after the crash.

2.4.4.31 Ex: Collision of two masses

Two balls A and B of different masses collide. A is initially at rest and B has the velocity v. After the shock, B has a speed of $v/2$ and moves perpendicular to the direction of the initial movement. Determine the direction of A's movement after the collision. What is the change in energy due to the collision?

2.4.4.32 Ex: Collision of two masses

A bullet of mass m is fired with the velocity v against a ballistic pendulum of mass M. The bullet passes through the pendulum and emerges with the velocity $v/4$.
a. calculate the maximum height of the oscillating pendulum,
b. calculate the energy dissipated when the bullet passes through the pendulum.

2.5 Further reading

H.M. Nussenzveig, Edgar Blucher (2013), *Curso de Física Básica: Mecânica - vol 1* [ISBN]

phet, *Interactive Simulations for Science and Math* [http]

sofisica, *Material de apoio didático* [http]

Chapter 3

Rotations and dynamics of rigid bodies

So far we have considered the translation dynamics of point masses. Obviously, point masses, unlike extended rigid bodies, cannot rotate.

3.1 Rotation about a fixed axis

Summary of common transformations,

operation	action on position	on momentum
translation	$\mathcal{T}_{\rm tr}\mathbf{r} = \mathbf{r} + \mathbf{a}$	$\mathcal{T}_{\rm tr}\mathbf{p} = \mathbf{p}$
kick	$\mathcal{T}_{\rm kc}\mathbf{r} = \mathbf{r}$	$\mathcal{T}_{\rm kc}\mathbf{p} = \mathbf{p} + m\mathbf{v}$
rotation	$\mathcal{T}_{\rm rt}\mathbf{r} = e^{\vec{\alpha}\times}\mathbf{r}$	$\mathcal{T}_{\rm rt}\mathbf{p} = e^{\vec{\alpha}\times}\mathbf{p}$
Galilei boost	$\mathcal{T}_{\rm G}\mathbf{r} = \mathbf{r} + \mathbf{v}t$	$\mathcal{T}_{\rm G}\mathbf{p} = \mathbf{p} + m\mathbf{v}$
transform to accelerated frame	$\mathcal{T}_{\rm ac}\mathbf{r} = \mathbf{r}$	$\mathcal{T}_{\rm ac}\mathbf{p} = \mathbf{p} + m\mathbf{g}t$
transform to rotating frame	$\mathcal{T}_{\rm ar}\mathbf{r} = e^{\vec{\omega}t\times}\mathbf{r}$	$\mathcal{T}_{\rm ar}\mathbf{p} = e^{\vec{\omega}t\times}\mathbf{p}$

Let us have a closer look at the rotation operator defined by,

$$\boxed{\mathcal{T}_{\rm rt}\mathbf{r} = e^{\vec{\alpha}\times}\mathbf{r}}\ . \tag{3.1}$$

It can be expanded as,

$$\begin{aligned}\mathcal{T}_{\rm rt}\mathbf{r} &= \sum_n \frac{(\vec{\alpha}\times)^n}{n!}\mathbf{r} = \mathbf{r} + \vec{\alpha}\times\mathbf{r} + \tfrac{1}{2}\vec{\alpha}\times(\vec{\alpha}\times\mathbf{r}) + ... \\ &= \hat{\mathbf{e}}_\alpha(\hat{\mathbf{e}}_\alpha\cdot\mathbf{r}) + \hat{\mathbf{e}}_\alpha\times\mathbf{r}\sin\alpha - \hat{\mathbf{e}}_\alpha\times(\hat{\mathbf{e}}_\alpha\times\mathbf{r})\cos\alpha\ .\end{aligned} \tag{3.2}$$

Choosing the rotation angle along Cartesian coordinates, we find matrix representations of the rotations transformations. For, $\vec{\alpha} = \alpha\hat{\mathbf{e}}_x$,

$$\mathcal{T}_{\rm rt}\mathbf{r} = x\hat{\mathbf{e}}_x + (y\hat{\mathbf{e}}_z - z\hat{\mathbf{e}}_y)\sin\alpha + (y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z)\cos\alpha = R_x\mathbf{r} \tag{3.3}$$

$$\text{with}\qquad R_x = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{pmatrix}\ ,$$

for, $\vec{\alpha} = \alpha\hat{\mathbf{e}}_y$,

$$\mathcal{T}_{\rm rt}\mathbf{r} = y\hat{\mathbf{e}}_y + (z\hat{\mathbf{e}}_x - x\hat{\mathbf{e}}_z)\sin\alpha + (x\hat{\mathbf{e}}_x + z\hat{\mathbf{e}}_z)\cos\alpha = R_y\mathbf{r} \tag{3.4}$$
$$\text{with} \qquad R_y = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} ,$$

and for, $\vec{\alpha} = \alpha\hat{\mathbf{e}}_z$,

$$\mathcal{T}_{\rm rt}\mathbf{r} = z\hat{\mathbf{e}}_z + (x\hat{\mathbf{e}}_y - y\hat{\mathbf{e}}_x)\sin\alpha - \hat{\mathbf{e}}_z \times (\hat{\mathbf{e}}_z \times \mathbf{r})\cos\alpha = R_z\mathbf{r} \tag{3.5}$$
$$\text{with} \qquad R_z = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Furthermore, since the vectors $\hat{\mathbf{e}}_\alpha$, $\hat{\mathbf{e}}_\alpha \times \mathbf{p}$, and $\hat{\mathbf{e}}_\alpha \times (\hat{\mathbf{e}}_\alpha \times \mathbf{p})$ in Eq. (3.2) obviously form an orthogonal coordinate system, we find defining $\gamma \equiv \angle(\hat{\mathbf{e}}_\alpha, \mathbf{p})$,

$$\begin{aligned} \mathcal{T}_{\rm rt}(e^{\vec{\alpha}\times}\mathbf{p})^2 &= [\hat{\mathbf{e}}_\alpha(\hat{\mathbf{e}}_\alpha \cdot \mathbf{p}) + \hat{\mathbf{e}}_\alpha \times \mathbf{p}\sin\alpha\hat{\mathbf{e}}_\alpha \times (\hat{\mathbf{e}}_\alpha \times \mathbf{p})\cos\alpha]^2 \\ &= [\hat{\mathbf{e}}_\alpha(\hat{\mathbf{e}}_\alpha \cdot \mathbf{p})]^2 + [\hat{\mathbf{e}}_\alpha \times \mathbf{p}\sin\alpha]^2 + [\hat{\mathbf{e}}_\alpha \times (\hat{\mathbf{e}}_\alpha \times \mathbf{p})\cos\alpha]^2 \\ &= \mathbf{p}^2\cos^2\gamma + \mathbf{p}^2\sin^2\alpha\sin^2\gamma + \mathbf{p}^2\cos^2\alpha\sin^2\gamma = \mathbf{p}^2 , \end{aligned} \tag{3.6}$$

that the kinetic energy and the potential energy in radial potentials is invariant to rotation,

$$\mathcal{T}_{\rm rt}\left(\frac{\mathbf{p}^2}{2m} + V(|\mathbf{r}|)\right) = \frac{\mathbf{p}^2}{2m} + V(|\mathbf{r}|) . \tag{3.7}$$

3.1.0.1 Infinitesimal rotations

For small rotation angles $\delta\vec{\alpha}$ the expression (3.2) can be written as,

$$\begin{aligned} \mathcal{T}_{\rm rt}\mathbf{r} &\simeq \mathbf{r} + \delta\vec{\alpha} \times \mathbf{r} \\ &\simeq \hat{\mathbf{e}}_\alpha(\hat{\mathbf{e}}_\alpha \cdot \mathbf{r}) + \hat{\mathbf{e}}_\alpha \times \mathbf{r}\delta\alpha - \hat{\mathbf{e}}_\alpha \times (\hat{\mathbf{e}}_\alpha \times \mathbf{r}) = \mathbf{r} + \hat{\mathbf{e}}_\alpha \times \mathbf{r}\delta\alpha . \end{aligned} \tag{3.8}$$

For example, for $\delta\vec{\alpha} = \delta\alpha\hat{\mathbf{e}}_z$, the expression (3.5) can be decomposed into,

$$\begin{aligned} \mathcal{T}_{\rm rt}(z)\mathbf{r} &= \left[\mathbb{I}_3 + \delta\alpha\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]\mathbf{r} = \begin{pmatrix} x - y\delta\alpha \\ y + x\delta\alpha \\ z \end{pmatrix} \\ &= \mathbf{r} + \hat{\mathbf{e}}_z \times \mathbf{r}\delta\alpha = \mathbf{r} + \frac{\partial\mathbf{r}}{\partial\alpha}\delta\alpha . \end{aligned} \tag{3.9}$$

Expressing (3.9) as a Taylor expansion,

$$\frac{\partial\mathbf{r}}{\partial\alpha} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{3.10}$$

3.1.1 Transformation into a rotating system

Transformation into a rotating system,

$$\frac{d}{dt}\mathcal{T}_{\text{ar}}\mathbf{r} = e^{\vec{\omega}t\times}\frac{d}{dt}\mathbf{r} + \left(\frac{d}{dt}e^{\vec{\omega}t\times}\right)\mathbf{r} = \left(\frac{d}{dt} + \vec{\omega}\times\right)\mathcal{T}_{\text{ar}}\mathbf{r} \ , \tag{3.11}$$

from which we deduce the rule,

$$\boxed{\left\{\frac{d}{dt}\right\}_{\text{lab}} = \left\{\frac{d}{dt}\right\}_{\text{rot}} + \vec{\omega}\times} \ . \tag{3.12}$$

Use this expression to resolve Exc. 3.1.2.1 to 3.1.2.4.

3.1.1.1 Inertial forces in the rotating system

Coriolis force, . Do the Excs. 3.1.2.5 to 3.1.2.14.

3.1.1.2 Inertial forces in the linearly accelerated system

3.1.2 Exercises

3.1.2.1 Ex: Rotating coordinate systems

A body of mass m is thrown horizontally at a velocity v_0 in the x-direction in the homogeneous gravitational field of the Earth. (Gravity towards $-z$). The rotation of the Earth is not taken into account.
a. Give the solution $\mathbf{r}(t)$ of the equation of motion.
b. Transform the solution into a coordinate system that rotates around the z-axis at the angular velocity ω.
c. Set up the equation of motion in the rotating system and show that the transformed path from (b) satisfies it.

3.1.2.2 Ex: Orbiting vulture

Consider two Cartesian coordinate systems that are identical at time $t = 0$. One rotates around the z-axis at a constant angular velocity ω, while the laboratory system is at rest. On the x-axis of this rotating system, a vulture moves from the coordinate origin (at time $t = 0$) at a constant speed v.
a. Calculate the vectors of the vulture's position and velocity in the rotating coordinate system.
b. Calculate the Cartesian coordinates of the vulture in the laboratory system and from this the velocity in the laboratory system.
c. Calculate directly the velocity observed in the laboratory system using the relationship (3.9).

3.1.2.3 Ex: Rotating system

Consider a rotating coordinate system, which at time $t = 0$ coincides with the lab system, but rotates relative to the lab system with angular velocity $\omega = 2\pi/T$, $T =$

10 s, around the z-axis. In this coordinate system the rotation of a mass point be given by the vector,

$$\mathbf{r}' = \begin{pmatrix} a \\ a \\ 0 \end{pmatrix} .$$

a. Determine the position vector of this point with respect to the laboratory coordinate system.
b. Consider the position vector at time $t = 5\,\mathrm{s}$ and determine its velocity measured in the rotating system and in the laboratory system.

3.1.2.4 Ex: Rotating system

Consider two coordinate systems, both with their origin at the center of the Earth and their z-axis parallel to the Earth's rotation axis. The rotating system rotates with the Earth, while the laboratory system is fixed to the solar system. An airplane moves at a constant velocity v relative to the Earth's surface on a direct trajectory from the north pole to the equator, where it arrives after a time $\tau = 12\,\mathrm{h}$. What is the velocity of the plane (as a function of time) seen by the lab system?

3.1.2.5 Ex: Inertial forces

A car travels at a speed $v = 100\,\mathrm{km/h}$ from São Carlos (latitude $\theta = -22^\circ$) to Ribeirão Preto, which is north of São Carlos.
a. Calculate the value of the Coriolis acceleration. In which direction is the car deviated?
b. Calculate the centrifugal force knowing that the radius of the Earth is 6370 km.

3.1.2.6 Ex: Foucault's pendulum

A pendulum of mass $m = 7\,\mathrm{kg}$ and length $L = 6\,\mathrm{m}$ is deflected by the angle $\beta = 4^\circ$ and then released. The experiment takes place at a latitude of $\Phi = 51^\circ$ north.
a. How strong is the Coriolis force when passing through the rest position?
b. What is the radius of curvature r of the projection of the pendulum's path onto the horizontal base at the rest point?
c. How long is the oscillation period T of a full rotation of the pendulum plane in the lab frame?

3.1.2.7 Ex: Earth rotation

a. A car with the mass $m = 1000\,\mathrm{kg}$ stands on the equator of the Earth. What is the total force acting on the car, and in which direction does it point? Assume the Earth to be a sphere with radius $R_{♁} = 6370\,\mathrm{km}$. The angular velocity is $\omega_{♁} = 7.27 \cdot 10^{-5}\,\mathrm{s}^{-1}$.
b. Now the car is traveling eastwards at a speed $v = 100\,\mathrm{km/h}$ along the equator. What total force is acting on it, and which direction does this force have?
c. How fast would the car have to travel eastwards along the equator to lift off the surface of the Earth?

Figure 3.1: Earth rotation.

3.1.2.8 Ex: Spinning top in a horizontal plane

Discuss the spinning top on a horizontal plane. What happens when the plane is inclined?

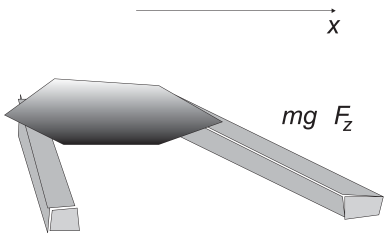


Figure 3.2: Spinning top.

3.1.2.9 Ex: Inertial forces

a. Calculate the correction for the weight of a mass m at rest at the equator due to centrifugal force. Does the centrifugal force modify the falling time for a mass released at a height above the Earth?
b. What are the value and orientation of the Coriolis acceleration acting on a body at the north pole moving with the speed $v = 100\,\text{m/s}$ to the south?
c. Calculate the deviation due to the Coriolis force for a body falling at the equator from a height of $h = 100\,\text{m}$.

3.1.2.10 Ex: Carousel

A vertical bar rotates around the z-axis at a constant angular velocity ω. A thread of length l is attached to the upper end of the rod, while a mass m is attached to the other end of the thread.
a. Find the tension that acts on the thread.
b. Find the angle between thread and bar in equilibrium.

3.1.2.11 Ex: Yo-yo

A yo-yo with radius R and thickness D is accelerated in the field of terrestrial attraction.
a. Determine the inertial moments with respect to the yo-yo's symmetry axis and to the instantaneous point of suspension on the string.
b. Calculate the torque and write down the equation for the rotational motion.
c. From the equation of motion, calculate the angular and linear accelerations, the angular and linear velocities, and the angular momentum.

3.1.2.12 Ex: Deviation to the east

An object is thrown up vertically at the initial speed v_0. Determine the east deflection based on the Earth's rotation as a function of latitude.

3.1.2.13 Ex: Coriolis force

A body falls in free fall at the equator of the Earth from a height of $h = 100\,\text{m}$. At the beginning of the movement ($t = 0$) the body rests at the height h. Air friction is neglected. The centrifugal acceleration is already taken into account in the value of the gravitational acceleration $g = 9.81\,\text{m/s}^{-2}$ relative to the Earth's surface.
a. Set up the equations of motion. Show that the path lies in a plane. What is its spatial location?
b. Solve the differential equations by combining the remaining components of $\mathbf{r}(t)$ into a complex variable $u(t)$. Pay attention to the initial conditions!
c. Give the solution $\mathbf{r}(t)$ and then approximate for small ωt (ω being the angular velocity of the Earth's rotation). Why is this approximation justified?
d. What is the distance from the real point of impact to the point that would be reached without the action of the Coriolis force? In which direction is the body deviated?

3.1.2.14 Ex: Coriolis force

A penguin with the mass m is located directly at the South Pole and begins to slide horizontally at the initial velocity v on the ice. The ice slows him down with the force $\mathbf{F} = -\gamma\mathbf{v}$.
a. Write down the equations of motion in a coordinate system originating at the south pole, which rotates with the Earth.
Note: Place the x-axis in the direction of the penguin's initial velocity and combine the variables x and y to a complex variable $z = x + \imath y$. Neglect the curvature of the Earth. b. Calculate the end position of the penguin on the assumption that the rotation speed of the earth ω is small enough to neglect quadratic terms $\propto \omega^2$ in the equations of motion or in their solution.

3.2 The rigid body

In mechanics a rigid body is defined as a system of masses whose mutual distances are kept fixed during any motion. The rigid bodies of practical interest are generally

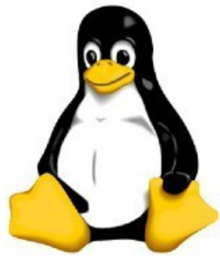


Figure 3.3: Coriolis force.

extended over macroscopic volumes and form a (quasi-)continuous mass distribution. The motion of large rigid bodies is more complicated than the motion of a point mass, since in addition to the translational motion, there may be rotations about one or more axes. The dynamics of both types of motion can clearly be separated in rigid bodies.

3.2.1 Translations and rotations: linear and angular momentum

The translational motion of a rigid body of mass M is fully described by the evolution of the coordinates and the velocity of its center-of-mass. In fact, one can assimilate the translation dynamics of the body with the whole mass M being concentrated in the center-of-mass. The total momentum $\mathbf{p}$ of the body is,

$$\mathbf{p} = m\mathbf{v} \ , \tag{3.13}$$

where $\mathbf{v}$ is the velocity of the center-of-mass. The equation that determines the translational dynamics is Newton's second law,

$$\mathbf{F}_{\text{ext}} = \frac{d\mathbf{p}}{dt} \ , \tag{3.14}$$

where $\mathbf{F}_{\text{ext}}$ is the vectorial sum of external forces acting on the body. When no external forces act, the amount of translational motion of the rigid body is conserved. Similarly, the translational kinetic and the gravitational potential energies of the rigid body can be evaluated by simply considering, respectively, the velocity and height H of the center-of-mass with respect to a reference level of potential energy:

$$E_{\text{kin}} = \frac{M}{2}v^2 \qquad \text{and} \qquad E_{\text{grv}} = MgH \ . \tag{3.15}$$

The (pure) rotation of a rigid body about its center-of-mass also contains kinetic energy. The quantity representing the amount of rotational motion of a rigid body is the rotational angular momentum. For simplicity, we will assume that the rotation occurs about an axis passing through the center-of-mass of the body, and that the body is symmetric around that axis. In this situation, the angular momentum reads,

$$\mathbf{L} = I\vec{\omega} \ , \tag{3.16}$$

where $\vec{\omega}$ is the angular velocity and I the moment of inertia of the rigid body with respect to the rotation axis. The moment of inertia is obtained by summing for the entire body over the contributions of the products between the elementary mass fragments δm_i and the square of their distances d_i^2 from the rotation axis,

$$I = \sum_i d_i^2 \delta m_i \ . \tag{3.17}$$

For an extensive body of volume V and density ρ, the sum in this equation is expressed as an integral along the infinitesimal mass elements $dm = \rho dV$,

$$I - \int_V d^2 \rho dV \ . \tag{3.18}$$

3.2.2 Rotational energy and moment of inertia

The equation that determines the dynamics of rotation is a consequence of Newton's second law, and results in,

$$\tau_{\text{ext}} = \frac{d\mathbf{L}}{dt} \ . \tag{3.19}$$

where τ_{ext} is the vectorial sum over the torques exerted by each external force acting on the body,

$$\tau_{\text{ext}} = \sum_i \mathbf{r}_i \times \mathbf{F}_{\text{ext},i} \ . \tag{3.20}$$

In this expression, $\mathbf{r}_i$ is the vector indicating the point of application of the force $\mathbf{F}_{\text{ext},i}$ on the body, measured with respect to the center-of-mass. When the total external torque is zero, the angular momentum of the rigid body is conserved. The kinetic energy associated with the rotation of the rigid body is given by the expression,

$$E_{\text{rot}} = \frac{I}{2}\omega^2 \ . \tag{3.21}$$

Do the Excs. 3.2.6.1 to 3.2.6.25.

Example 6 (*Cylinder on a slope*): A cylinder of radius R rolls down a slope of inclination θ. We have $h = z\sin\theta$ and $v = R\omega$. Energy conservation means, $mgh = \frac{m}{2}v^2 + \frac{I}{2}\omega^2$, hence,

$$v = \sqrt{\frac{2gh}{1 + I/mR^2}} \ . \tag{3.22}$$

For a plain cylinder we know $I = \frac{m}{2}R^2$ and for a hollow one we know $I = \frac{m}{2}(R^2+r^2)$. This means, plain cylinders will always reach the same final velocity,

$$v = \sqrt{\frac{4gh}{3}} \ , \tag{3.23}$$

while hollow cylinders will be slower,

$$v = \sqrt{\frac{4gh}{3 + r^2/R^2}} \ . \tag{3.24}$$

3.2.3 Rotation dynamics about a fixed axis

3.2.4 Static equilibrium of a rigid body

3.2.5 Constant acceleration

3.2.6 Exercises

3.2.6.1 Ex: Center of mass

You have three rectangular blocks of same heights and depths, but with different lengths and different masses. You want to build an inclined tower with them. Determine experimentally or theoretically how to stack the blocks so that the top block overhangs as much as possible. What is the most favorable order in the case of same masses but different lengths? What is the most advantageous order in the case of same lengths but different masses?

3.2.6.2 Ex: Inclined pyramid

Five identical point masses m are arranged at the corners of an oblique pyramid with the height H and a square base area $F = a^2$, i.e. they are located at $(x, y, z) = (\pm a/2, \pm a/2, 0)$ and $(b, 0, H)$. What is the maximum lateral displacement b of the top of the pyramid so that the pyramid does not tip out of the horizontal xy-plane?

3.2.6.3 Ex: Inclined cone

A rigid cone with a homogeneous mass distribution has the height H and a base area with a radius R. How far can you tilt the cone before it falls over?

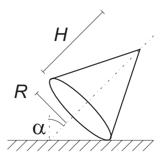


Figure 3.4: Inclined cone.

3.2.6.4 Ex: Angular momentum of the Earth

a. Calculate the angular momentum of the Earth's rotation around its center of mass (mass $M = 5.97 \times 10^{24}$ kg, radius $R = 6370$ km).
b. Compare this with the angular momentum of the Earth's rotation around the sun (distance between Sun and Earth $D = 150 \times 10^9$ km).

3.2.6.5 Ex: Inertial moment

a. Calculate the mass of a cone with homogeneous density (height H and a base radius R).
b. Calculate the cone's inertial moment with respect to the symmetry axis.

c. The cone now rotates with the frequency ω around an axis displaced by a distance R parallel to the symmetry axis. How to calculate the angular momentum?

3.2.6.6 Ex: Glass of beer

A cylindrical glass of beer (height $H = 18\,\text{cm}$, diameter $D = 6\,\text{cm}$, mass $m_G = 0.5\,\text{kg}$, the bottom is without mass) has, when empty, its center of mass exactly halfway up. Obviously, it is there too, when the glass is filled with beer (density $\rho = 1000\,\text{kg/m}^3$) up to the upper limit. Down to which height h do you have to drink the beer for the center of mass to be at its lowest height?

3.2.6.7 Ex: Inertial moment

Consider a flat symmetrical triangle with surface density σ and corner length a. Fix the coordinate system such that the triangle is given by the corners $A = (0, \frac{a}{2})$, $B = (0, -\frac{a}{2})$ e $C = (\frac{a\sqrt{3}}{2}, 0)$.
a. Calculate the mass of the triangle.
b. Calculate the center of mass.
c. Calculate the inertial moment with respect to an axis normal to the surface through the center of mass of the triangle.

3.2.6.8 Ex: Inertial moment

Consider a flat arc segment with surface density σ, radius R, and angle ϕ. Fix the coordinate system at the center of the arc such that the axis of symmetry is x.
a. Calculate the mass of the segment.
b. Calculate the center of mass.
c. Calculate the inertial moment with respect to a normal axis at the surface through the center of mass of the segment.

3.2.6.9 Ex: Inertial tensor

Calculate the moment of inertia of a homogeneous hollow cylinder with mass m, length L, outer radius R, and inner radius r.

3.2.6.10 Ex: Inertial tensor

Determine the tensor of inertia for
a. a tetrahedron in which the mass is distributed equally between the corner points in a coordinate system in which the z-axis passes through the center of opposite edges;
b. of the tetrahedron in a coordinate system in which the z-axis passes through one of its corners and through its center of gravity.

3.2.6.11 Ex: Inertial tensor

Two balls of different radii $R_1 > R_2$ and same homogeneous mass density ρ are welded together at their point of contact. Derive the moment of inertia of each individual sphere (axis of rotation through the center).
a. Where is the center-of-mass of the total system?

b. Calculate the main moments of inertia of the total system with regard to its center of gravity. **Note:** Use Steiner's theorem.

3.2.6.12 Ex: Rolling movement

A cylinder of mass m with radius R rolls over the ski jump shown in the figure, which has a gradient of -173% (i.e. $\tan\alpha = -1.73$). After having left the ski jump at a height of $h = H/2$, he falls in the Earth's gravity field. Compare the flight distance with the case in which the cylinder does not roll, but slides smoothly over the hill.

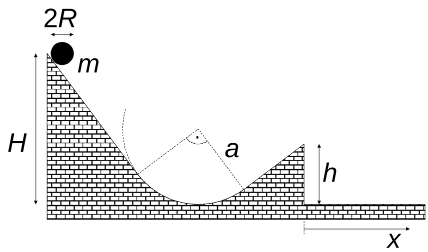


Figure 3.5: Rolling movement.

3.2.6.13 Ex: Accelerated rotational movement

A mass $m = 1\,\mathrm{kg}$ is hanging on a light cord, which is wound on a $d = 2\,\mathrm{cm}$ thick wheel with the radius $r = 10\,\mathrm{cm}$. The wheel has a homogeneously distributed mass of $M = 10\,\mathrm{kg}$ and rotates without friction.
a. Determine the tensile force in the cord.
b. Determine the acceleration of the mass.

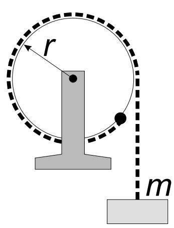


Figure 3.6: Accelerated rotational movement.

3.2.6.14 Ex: Hovering dumbbell

A dumbbell rotates at the angular velocity ω around a rotation axis that passes through its center of gravity and forms an angle α with the axis connecting the masses (see figure).
a. Calculate the angular momentum vector of the barbell in the co-rotating coordinate system.
b. How does the angular momentum behave in a non-rotating coordinate system?

c. Calculate the torque that must act so that the angular velocity remains constant.

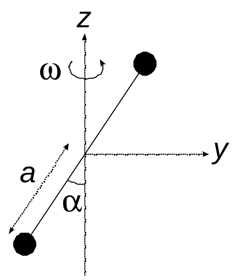


Figure 3.7: Hovering dumbbell.

3.2.6.15 Ex: Hovering cross

Four masses of equal weight m are connected to each other in the manner shown by a rigid cross. The mass of the connecting rods is negligible. The cross rotates at a constant angular velocity ω around an axis of rotation that lies in its plane, passes through its center of gravity and forms an angle α with its long axis.
a. Calculate the inertia tensor I of the body in the co-rotating system. b. First

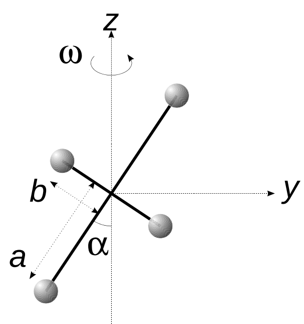


Figure 3.8: Hovering cross.

calculate the angular momentum **L** in the co-rotating system using the relationship $\mathbf{L} = I\vec{\omega}$, where $\vec{\omega} = (0, 0, \omega)$ is invariant under rotation around the z-axis. Then transform **L** back into the laboratory system.
c. Calculate the torque that must act so that the angular velocity remains constant.

3.2.6.16 Ex: A farmer on a ladder

A farmer of mass $M = 90\,\text{kg}$ harvests apples climbing a ladder (length $L = 10\,\text{m}$, mass $m = 30\,\text{kg}$) leaning at an angle $\phi = 60°$ against a branch relative to the Earth's surface. While harvesting the highest apples at the top, the branch suddenly breaks away. In his fear, the farmer clings to the ladder. The system ladder-farmer now

falls down with the ladder base remaining fixed. Neglect the spatial expansion of the farmer. The ladder can be viewed as a one-dimensional rod.
a. Show that the moment of inertia I_{LB} of the ladder-farmer system for a rotation around the ladder base point can be given as $I_{\mathrm{LB}} = \left(\frac{m}{3} + M\right) L^2$.
b. At what speed v_1 does the farmer reach the surface of the Earth?
c. At the instant when the branches break, in his shock the farmer drops an apple. At what speed v_2 does the apple reach the surface of the Earth?
d. Would the farmer fall more slowly when letting the ladder go at the moment when the branch breaks?

3.2.6.17 Ex: Angular momentum

A thin bar of length L and mass M is suspended at one end thus forming a physical pendulum.
a. I calculate the inertial moment of the bar.
b. Now, the pendulum is tilted a little. Calculate the torque as a function of the angle of inclination.
c. For very small angles θ holds the approximation $\sin\theta \simeq \theta$. Use this approach to establish the equation of motion.

3.2.6.18 Ex: Accelerated pendulum

A pendulum of length R is suspended in a car which is uniformly accelerated with the acceleration a.
a. Calculate the angle of the pendulum's displacement.
b. Calculate the torque exerted by the accelerating force of the car.
c. What is the torque exerted by the weight in this situation?
d. Calculates the ratio of the torques. Justify the obtained result.

3.2.6.19 Ex: Rotating disc

A round disk of mass M with radius R is rotatably mounted about a vertical axis passing through its center. A spring of negligible mass with the spring constant κ is attached to the disc tangentially to the edge. The spring, which is initially fully compressed by the length d, represents a launching device for a ball of mass m. The disk rests until the ball is released being ejected horizontally by the spring. Calculate the angular velocity of the disk after the ejection.
Help: Use the angular momentum and energy conservation laws.

3.2.6.20 Ex: Billiard

A resting billiard ball with radius r is played with a horizontal queue, which gives it an impulse $\Delta p = F\Delta t$. The queue hits the ball at a height h above the table. How does the initial angular velocity ω of the sphere depend on Δp and h? **Help:** The ball rolls without slipping.

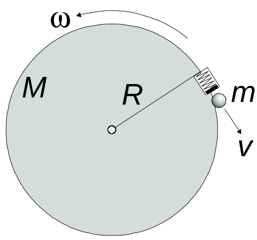


Figure 3.9: Rotating disc.

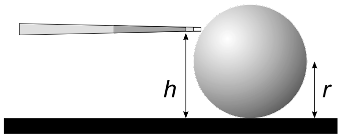


Figure 3.10: Billiard.

3.2.6.21 Ex: Rotating and propagating arc

Calculate the angular momentum of an arc (a) rotating around its center of mass, (b) rotating around a peripheral point, and (c) rolling without friction on a surface.

3.2.6.22 Ex: Inertial momentum

Calculate the moment of inertia of a quadrilateral of point masses in relation to the axes shown in the figure.

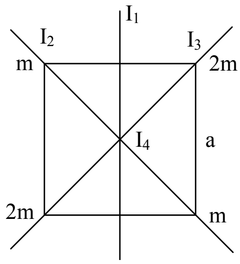


Figure 3.11: Inertial momentum.

3.2.6.23 Ex: Inertial momentum

A disc of radius R and surface mass density σ has a circular hole of radius r at a distance a from the center of the disc. Calculate the moments of inertia with respect to the axes 1, 2 and 3, as shown in the figure.

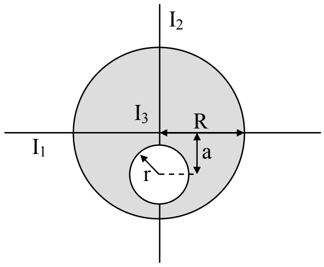


Figure 3.12: Inertial momentum.

3.2.6.24 Ex: Inertial momentum

Calculate the moment of inertia of a sphere of mass M and radius R with respect to an axis passing through the center of mass.

3.2.6.25 Ex: Inertial momentum of a slender bar

A thin bar of mass M and length L makes an angle θ with the y-axis, as shown in the figure.
a. Calculate the moment of inertia for rotation about the axis;
b. Calculate the moment of inertia for rotation around an axis parallel to y and passing through the center of mass.

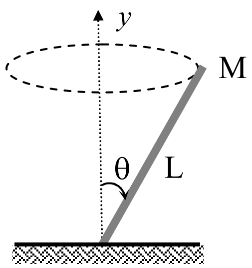


Figure 3.13: Slender bar.

3.3 Angular momentum

3.3.1 Torque and angular momentum of a particle system

3.3.1.1 Steiner's theorem

The angular momentum I_ω of a rigid body with respect to a rotation axis $\hat{\mathbf{e}}_\omega$ can be divided into $I_\omega = I_\omega^{\text{SP}} + Mb^2$. Here, I_ω^{SP} is the inertial moment of the body with respect to the rotation axis $\hat{\mathbf{e}}_{0\omega}$, which is parallel to $\hat{\mathbf{e}}_\omega$ and traverses the center-of-mass of the body, M is the mass of the body, and b is the distance from the rotation axis. This is *Steiner's theorem*.

To demonstrate Steiner's theorem, we consider the vector of a mass point α of the rigid body a first time in the coordinate system, whose origin lies on the axis $\hat{\mathbf{e}}_\omega$, and which we call $\mathbf{r}_\alpha$. A second time we describe this mass point in the coordinate system, whose origin lies in the center-of-mass, and which we call $\mathbf{r}$. The vector of the position of the mass α in the center-of-mass system be described by ρ_α. With this nomenclature,

$$\mathbf{r}_\alpha = \mathbf{r} + \vec{\rho}_\alpha \ .$$

The moment of inertia with respect to the $\hat{\mathbf{e}}_\omega$-axis is then,

$$\begin{aligned}
I_\omega &= \sum_\alpha m_\alpha \left(\mathbf{r}_\alpha^2 - (\mathbf{r}_\alpha \cdot \hat{\mathbf{e}}_\omega)\right) \\
&= \sum_\alpha m_\alpha \left[(\mathbf{r} + \vec{\rho}_\alpha)^2 - ((\mathbf{r} + \vec{\rho}_\alpha) \cdot \hat{\mathbf{e}}_\omega)^2\right] \\
&= \sum_\alpha m_\alpha \left[(\mathbf{r}^2 + 2\mathbf{r}\vec{\rho}_\alpha + \vec{\rho}_\alpha^2 - (\mathbf{r} \cdot \hat{\mathbf{e}}_\omega)^2 - 2(\mathbf{r} \cdot \hat{\mathbf{e}}_\omega)(\vec{\rho}_\alpha \cdot \hat{\mathbf{e}}_\omega) - (\vec{\rho}_\alpha \cdot \hat{\mathbf{e}}_\omega)^2\right] \\
&= MR^2 + 0 + \sum_\alpha m_\alpha \vec{\rho}_\alpha^2 - M(\mathbf{r} \cdot \hat{\mathbf{e}}_\omega)^2 - 0 - \sum_\alpha m_\alpha \left(\vec{\rho}_\alpha \cdot \hat{\mathbf{e}}_\omega\right)^2 \\
&= I_\omega^{\text{SP}} + Mb^2 \ .
\end{aligned}$$

The zeros of the fourth line come from $\sum_\alpha m_\alpha \vec{\rho}_\alpha = 0$, because the left side just corresponds to the center-of-mass vector multiplied with the total mass, but which has in the center-of-mass system the value $\mathbf{r} = 0$.

3.3.2 Rotational work-energy relation

3.3.3 Conservation of angular momentum

Like the conservation law for linear momentum, the conservation law for angular momentum is universal. It holds for a rotational collision,

$$(I_1 + I_2)\omega = I_1\omega_1 + I_2\omega_2 \ . \tag{3.25}$$

On the other side, non-rotational energy is not conserved in a rotational collision, $E_{\text{rot}} < E_{\text{rot},1} + E_{\text{rot},2}$, because part of the energy can be dissipated:

$$E_{\text{rot},1} + E_{\text{rot},2} - E_{\text{rot}} = \frac{I_1}{2}\omega_1^2 + \frac{I_2}{2}\omega_2^2 - \frac{I_1 + I_2}{2}\omega^2 = \frac{-I_1 I_2}{2(I_1 + I_2)}(\omega_1 - \omega_2)^2 \ . \tag{3.26}$$

3.3.4 Combination of translation and rotation

Do the Excs. 3.3.5.1 to 3.3.5.20.

Example 7 (*Gyroscope*): The angular momentum due to the rotation of a wheel is,

$$\mathbf{L}_\omega = I_\omega^{(cm)}\vec{\omega} \ .$$

for a disk-shaped wheel the inertial moment $I_\omega^{(cm)}$ about an axis of rotation $\vec{\omega}$ crossing the center-of-mass is,

$$I_\omega^{(cm)} = \frac{M}{2}R^2 \ .$$

The force of gravitation,

$$\mathbf{F}_g = m\mathbf{g}$$

produces a torque,

$$\vec{\tau}_g = \mathbf{D} \times \mathbf{F}_g = Dmg\hat{\mathbf{e}}_\phi \sin\theta \ .$$

Initially, the rotation axis of the disk is horizontal $\theta = 0$. Due to the torque the shaft tilts downward thus forcing the angular momentum to shift to $\mathbf{L}_\omega + d\mathbf{L}_\omega$ with the velocity,

$$\tfrac{d\mathbf{L}_\omega}{dt} = \vec{\tau}_g \ .$$

How can the system react to compensate for this change of $\mathbf{L}_\omega$?
The *total angular momentum in the direction* $\mathbf{g}$ *must be conserved*, as there is no external torque in it. Fortunately, there is another possible movement that can generate an angular momentum: the rotation around the point of support O with the angular velocity $\vec{\Omega}$:

$$const = \mathbf{L}_{\rm tot} = \mathbf{L}_\omega + \mathbf{L}_\Omega \qquad \text{or} \qquad \frac{d\mathbf{L}_\omega}{dt} = -\frac{d\mathbf{L}_\Omega}{dt} \ .$$

That is, we have a torque $\vec{\tau}_\Omega = \frac{d\mathbf{L}_\Omega}{dt}$. Hence, $|\tau_\omega| = |\tau_\Omega|$, but since the orientation of $\mathbf{L}_\Omega$ is vertically fixed, the orientation of $\vec{\tau}_\Omega$ is also vertical, which corresponds to an azimuthal force. Thus, instead of an inclination of the rotation axis $\mathbf{L}_\omega$, we get an *azimuthal* shift by an angle,

$$d\vec{\phi} = \frac{d\mathbf{L}_\Omega}{L_\omega \sin\theta} = \frac{d\mathbf{L}_\omega}{L_\omega \sin\theta} = \frac{\vec{\tau}_g dt}{I_\omega^{(\rm cm)}\omega\sin\theta} = \frac{Dm(-\mathbf{g})\sin\theta dt}{I_\omega^{(\rm cm)}\omega\sin\theta} = -\frac{Dm\mathbf{g}dt}{I_\omega^{(\rm cm)}\omega} \ .$$

The frequency of this *precession* movement,

$$\vec{\Omega} = \frac{d\vec{\phi}}{dt} \ ,$$

produces an angular momentum,

$$\mathbf{L}_\Omega = M\mathbf{D} \times \mathbf{v}_\Omega = MD^2\vec{\Omega} = I_\Omega^{(\rm O)}\vec{\Omega} \ .$$

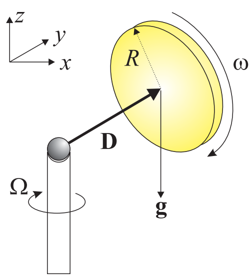


Figure 3.14: Rotating wheel supported on one end of its axis.

Now, the precession modifies the direction of the angular momentum in the plane,

$$\mathbf{L}_\omega = L_\omega \hat{\mathbf{e}}_\phi \ ,$$

with $\hat{\mathbf{e}}_\phi = \hat{\mathbf{e}}_x \cos\Omega t + \hat{\mathbf{e}}_y \sin\Omega t$. Hence,

$$\dot{\mathbf{L}} = L\vec{\Omega} = \vec{\tau}_\omega = L\frac{-Dm\mathbf{g}dt}{I_\omega^{(\rm cm)}\omega} = -Dm\mathbf{g} = \mathbf{D} \times \mathbf{F} \ .$$

Finally, we find a force exactly compensating gravity,

$$\mathbf{F} = -m\mathbf{g} \; .$$

3.3.5 Exercises

3.3.5.1 Ex: Statics of a ladder

A ladder of mass M and length L leans on a frictionless wall and and stands on a floor with friction μ (see figure). Knowing that the angle between the ladder and the wall is 45°, what should be the force on a rope tied in the middle of the stairs so that it does not fall?

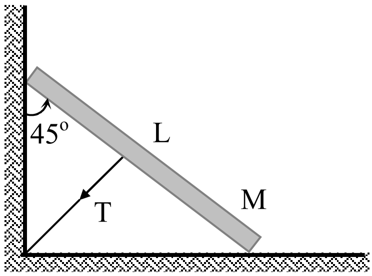


Figure 3.15: Statics of a ladder.

3.3.5.2 Ex: Statics of a ladder

A ladder of mass M and length L leans against wall and stands on the floor (both without friction) such as to form an angle θ with the wall, as shown in the figure. A rope tied at a height of H (parallel to the floor) keeps the ladder at rest. Calculate:
a. the tension in the rope;
b. the maximum height $H_{\max}$ at which equilibrium is possible;
c. the angular acceleration at the instant this rope is cut.

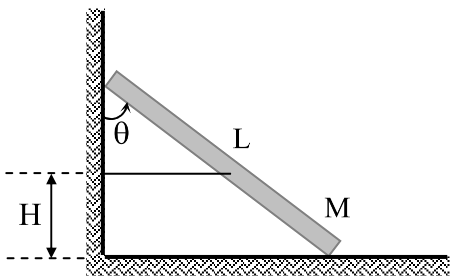


Figure 3.16: Statics of a ladder.

3.3.5.3 Ex: Statics of a ladder

A Λ-shaped ladder of mass $2M$ mass is opened to form an angle θ. What should be the coefficient of static friction with the floor so that it doesn't fall? (see figure).

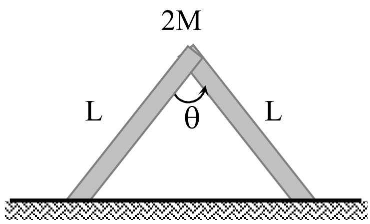


Figure 3.17: Statics of a ladder.

3.3.5.4 Ex: Statics of a ladder

A painter of mass M stands at the top of a Λ-shaped ladder of negligible weight (length of each side: L) that rests on an extremely smooth floor M. There is a crossbar at half height that prevents the ladder from opening. The vertex angle is θ. What is the force on the crossbar?

3.3.5.5 Ex: Statics of a ladder

A bar of length L and mass M is placed over a hole, as shown in the figure. What must be the friction coefficient for the bar to stay at rest?

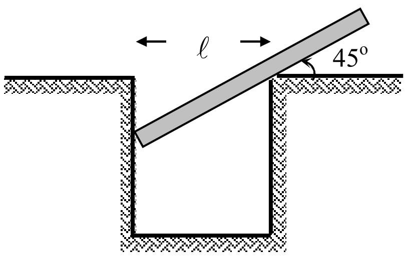


Figure 3.18: Statics of a ladder.

3.3.5.6 Ex: Torque on a block

On a smooth surface, a cubic block of size L and mass M slides with speed v (see figure). At a certain point, the cube hits a small obstacle. How fast should the velocity v of the block be in order to rotate around this point?

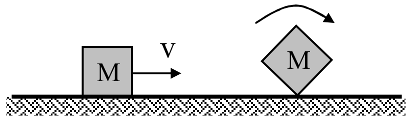


Figure 3.19: Torque on a block.

3.3.5.7 Ex: Falling rod

At the end of a rod of length L the negligible mass is placed a mass M. The system is released from vertical under the action of gravity. What is the equation that describes the angle $\theta(t)$ (see the figure)?

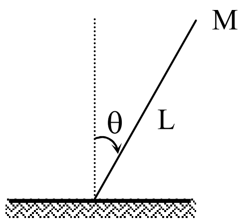


Figure 3.20: Falling rod.

3.3.5.8 Ex: Rotating arc

An arc of radius R, which rotates with angular velocity ω_0, is placed on a rough horizontal surface, as shown in the figure, the speed of its center of mass being zero. Determine the speed of the center of mass after slipping has ceased.

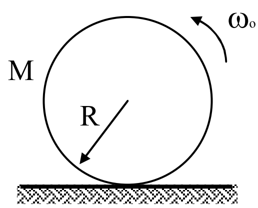


Figure 3.21: Rotating arc.

3.3.5.9 Ex: Angular momentum

The integral of the torque with respect to time is called the angular momentum. Starting from the relationship $\vec{\tau} = d\mathbf{L}/dt$, show that the impulse is the variation of the angular momentum.

3.3.5.10 Ex: Billiard

A billiard ball initially at rest receives an instant boost from a cue. The cue is kept horizontal at a distance h from the center. The ball leaves with speed v_0 and the final speed is $9v_0/7$. Show that $h = 4R/5$, where R is the radius of the sphere.

3.3.5.11 Ex: Bohr's atom

Niels Bohr postulated that a rotating mechanical system can only have angular momentum with multiple values of a constant $\hbar$, called Planck's constant $\hbar = h/2\pi = 1.054 \times 10^{-34}$ Js, that is: $L = I\omega = n\hbar$, being n a positive integer or zero.
a. Show that with this postulate, the energy of a rotor can only acquire discrete, that is, quantized values.
b. Consider a mass m forced to rotate on a circle of radius R (e.g. hydrogen atom). What are the possible values for the angular velocity considering Bohr's postulate?
c. What kinetic energy values can the atom adopt?

3.3.5.12 Ex: Earth rotation

Many of the great rivers flow into the equatorial region carrying sandy sediments. What effect does this have on the Earth's rotation?

3.3.5.13 Ex: Cylinder rolling on an inclined plane

A cylinder of mass M and radius R rotates without sliding on a horizontal plane. The speed of the center of mass is v. It encounters a plane with a tilt angle θ in front of it, as shown in the figure.
b. What were the linear and angular momenta when the cylinder meets the inclined plane?
c. How long does the cylinder take to reach the maximum height, and to what height does it rise on the inclined plane?
d. In this position, what was the change in angular momentum?

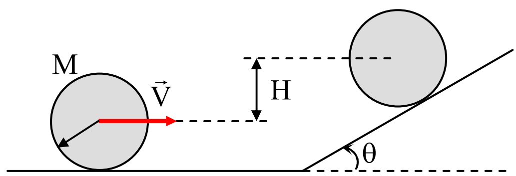


Figure 3.22: Cylinder rolling on an inclined plane.

3.3.5.14 Ex: Disc pushed by a mass

A disk of mass M and radius R can move around an axis passing through its center of mass, as shown in the figure. A particle of mass M as well follows a linear path with velocity v_i and impact parameter $d = R/2$ relative to the center-of-mass point. When it hits the disk it undergoes a deflection of 90° and has its velocity changed to

$v_f = v_i/2$.
a. What is the angular velocity of the disc after the collision?
b. What is the energy dissipated in the collision?

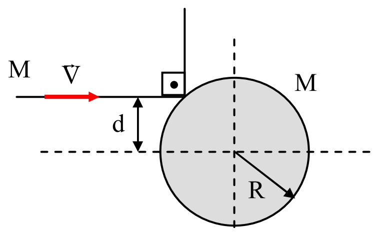


Figure 3.23: Disc pushed by a mass.

3.3.5.15 Ex: Disc pushed by a mass

A disc of mass $2m$ and radius R rests on an extremely smooth horizontal table. A bullet of mass m, speed v_0 and impact parameter R hits the disc and chokes it (see figure). Calculate:
a. The angular velocity of the system right after the collision;
b. The center-of-mass velocity after the collision;
c. The energy dissipated in the collision.

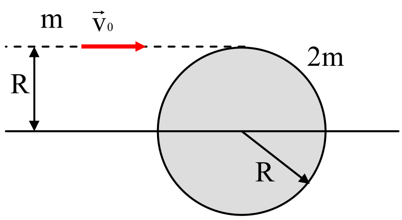


Figure 3.24: Disc pushed by a mass.

3.3.5.16 Ex: Billiard

A billiard ball initially at rest receives an sudden impulse from a cue, which forms an angle with the horizontal, as shown in the figure. The ball leaves with initial velocity v_0 and at the end of the movement it is at rest.
a. Determine the angle θ for this to happen.
b. What is the initial angular velocity of the ball?
c. What is the energy dissipated during the movement?

3.3.5.17 Ex: Horizontal pendulum with variable length

A particle of mass m is attached to the end of a wire and follows a circular path of radius r on a horizontal table without friction. The wire passes through a hole in the

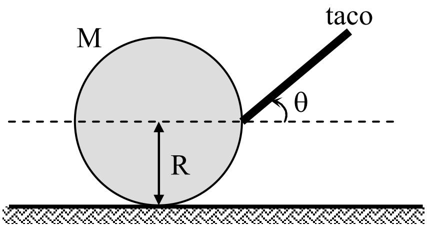


Figure 3.25: Billiard.

table and the other end is initially fixed. In this situation, the initial radius is r_0 and the initial angular velocity is ω_0. The wire is then slowly pulled in order to reduce the radius of the circular path, as shown in the figure.
a. How will the angular velocity vary as a function of r?
b. What work has to be done to bring the particle to the radius $r_0/2$?

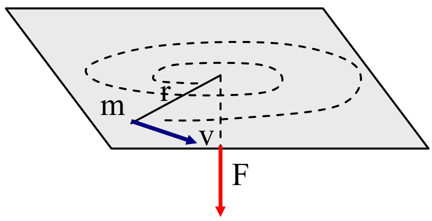


Figure 3.26: Horizontal pendulum with variable length.

3.3.5.18 Ex: Cylinder rolling on an inclined plane

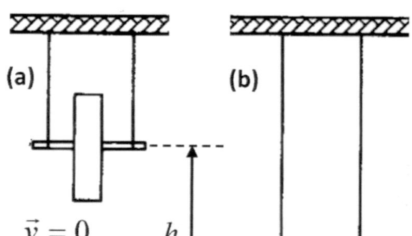
Consider a cylinder of mass M and radius R descending an inclined plane of angle θ without sliding. Calculate the acceleration of the center of mass and the friction force acting on the cylinder.

3.3.5.19 Ex: Billiard

A billiard ball of mass M and radius R ($I = \frac{2}{5}MR^2$ slides without rotating with speed v_0 on a frictionless table. Suddenly it encounters a part of table with friction and after some time is running without sliding.
a. Calculate the final speed of the ball.
b. What is the energy dissipated in the process?

3.3.5.20 Ex: Shock and conservation laws

In an experiment Maxwell's wheel was used to determine the moment of inertia I of a body of mass $m = 1490\,\text{g}$. The axis of the body around



which it spins during the fall has a radius of $r = 0.6\,\text{cm}$ (the mass of the axis is included in the mass m of the body). See figure below. Measurements of the fall time t_b were made for various heights h, and the data are given in the table below. The moment of inertia I is given by the expression,

$$I = \left(\frac{gt_b^2}{2h} - 1\right) mr^2 \ ,$$

which implies the following relationship between h and t_b,

$$h = \alpha t_b^2 \qquad \text{where} \qquad \alpha = \frac{g}{2}\frac{mr^2}{I + mr^2} \ .$$

a. Using a double logarithmic paper draw the graph $\log h \times \log t_b$. Choose the line that best approximates the data and determine its angular and linear coefficients. From them determine (i) The power law of t_b in the expression $h = \alpha t_b^\beta$ and (ii) The value of α and from it the value of I.
b. Using a millimetric paper draw the graph $h \times t_b^2$, and from the slope α determine the value of I again. Compare the results.

h (cm)	t_b (s)
10	1.2
20	1.8
30	2.1
40	2.5
50	2.8
60	3.0
70	3.3
80	3.6
90	3.7

3.4 Further reading

H.M. Nussenzveig, Edgar Blucher (2013), *Curso de Física Básica: Mecânica - vol 1* [ISBN]

Chapter 4

Vibrations

Vibrations are periodic processes, that is, processes that repeat themselves after a given time interval. After a time called *period*, the system under consideration returns to the same state in which it was initially. There innumerable examples for periodic processes, such as the motion of a seesaw, oceanic tides, electronic $L - C$ circuits, alternating current or rotations like that of the Earth around the Sun. Thus, vibrations are among the most fundamental processes in all domains of physics. A lecture version of this chapter can be found at (watch talk).

4.1 Free periodic motion

A movement is considered as free, when apart from a *restoring force*, that is a force working to counteract the displacement, there are no other forces accelerating or slowing down the motion.

4.1.1 Clocks

Periodic motions are used to *measure time*. Assuming a given process to be truly periodic, we can inversely *postulate* that the time interval within which this process occurs is constant. This interval is used to define a *unit of time*. For example, the 'day' is defined as the interval that the Earth needs to complete a rotation about its axis. The 'second' is defined as the 86400-th fraction of this period. Taking the second inversely as the base unit, we can define the day as the time interval needed for a periodic process taking 1 s to occur 86400 times. That is, we count the number of times ν that this process occurs within a day and calculate the duration of a day through,

$$\Delta T = \frac{1}{\nu} \ . \tag{4.1}$$

In real life, vibrations are subject to perturbations, just like all physical processes. These perturbations may afflict the periodicity and falsify the measurement of time. For example, the oceanic tides, which depend on the rotation of the moon around the Earth, can influence the Earth's own rotation. One of the challenges of *metrology*, which is the science dealing with issues related to the measurement of time, is to identify processes in nature that are likely to be insensitive to external perturbations. Nowadays, the most stable known periodic processes are vibrations of electrons within atoms. Therefore, the international time is defined by an atomic clock based on cesium: The 'official' second is the time interval in which the state of an electron oscillates 9192631770 times when the hyperfine structure of a cesium atom is excited by a microwave.

The unit of time is,

$$\text{unit}(T) = \text{s} \ . \tag{4.2}$$

A *frequency* is defined as the number of processes that occur within one second. We use the unit,

$$\text{unit}(\nu) = \text{Hz} \ . \tag{4.3}$$

Often, to simplify mathematical formulas, we will use the derived quantity of the *angular frequency* also called *angular velocity*,

$$\omega \equiv 2\pi\nu \ . \tag{4.4}$$

It has the unit,

$$\text{unit}(\omega) = \text{rad/s} \neq \text{Hz} \ . \tag{4.5}$$

It is important not to use the unit 'Hertz' for angular frequencies in order to avoid confusion.

4.1.2 Periodic trajectories

Many periodic processes are based on repetitive trajectories of particles or bodies. As an example, let us the movement of a body in a box shown in Fig. 4.1. When the body encounters a wall, it is elastically reflected thereby maintaining its velocity but reversing the direction of propagation. Clearly, the velocity is the derivative of the position,

$$v(t) = \dot{x}(t) \ . \tag{4.6}$$

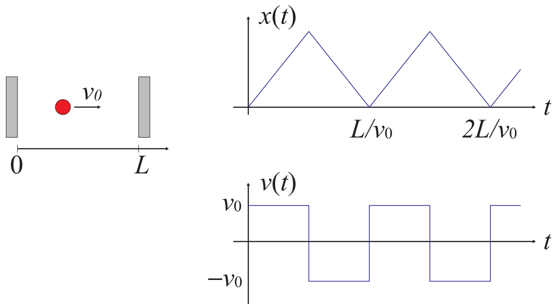


Figure 4.1: Trajectory of a body in a rectangular box. Upper trace: instantaneous position. Lower trace: instantaneous velocity.

To fully describe the trajectory of a body and to identify, when the trajectory repeats, two parameters are needed. Specifying, for example, the time evolution of position $x(t)$ and velocity $v(t)$, we can search for time intervals T after which,

$$x(t_0 + T) = x(t_0) \qquad \text{and} \qquad v(t_0 + T) = v(t_0) \ . \tag{4.7}$$

Obviously, as seen in Fig. 4.1, it is not enough just to look for the time when $x(t_0+T) = x(t_0)$.

4.1.3 Simple harmonic motion

The simplest motion imaginable is the harmonic oscillation described by,

$$x(t) = A\cos(\omega_0 t - \phi) \ , \tag{4.8}$$

and exhibit in Fig. 4.2. A is the *amplitude* of the motion, such that $2A$ is the distance between the two turning points. $T = 2\pi/\omega_0$ is the oscillation period, since,

$$\cos[\omega_0(t+T) - \phi] = \cos[\omega_0 t + 2\pi - \phi] = \cos[\omega_0 t - \phi] \ . \tag{4.9}$$

ϕ is a *phase shift* describing the time delay $t = \phi/\omega_0$ for the oscillation to reach the turning point.

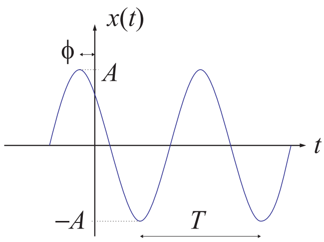


Figure 4.2: Illustration of the cosenus function with the amplitude A, the period T and the phase being negative for this graph $\phi < 0$.

The velocity and acceleration follow from,

$$v(t) = \dot{x}(t) = -\omega_0 A\sin(\omega_0 t - \phi) \qquad \text{and} \qquad a(t) = \dot{v}(t) = -\omega_0^2 A\cos(\omega_0 t - \phi) \ . \tag{4.10}$$

with this we can, using Newton's law, calculate the force necessary to sustain the oscillation of the body,

$$F(t) = ma(t) = -m\omega_0^2 A\cos(\omega_0 t - \phi) = -m\omega_0^2 x(t) \equiv kx(t) \ . \tag{4.11}$$

That is, in the presence of a force, which is proportional to the displacement but with the opposite direction, $F \propto -x$, we expect a sinusoidal solution. The proportionality constant k is called *spring constant*. Obviously the oscillation frequency is independent of amplitude and phase,

$$\omega_0 = \sqrt{k/m} \ . \tag{4.12}$$

Solve Exc. 4.1.10.1 and 4.1.10.2.

Example 8 (*Harmonic vibration*):

- Suspended spring-mass system, pendulums with various masses and lengths of wire, oscilloscope and function generator, water recipient with a floating body.

4.1.4 The spring-mass system

Let us now discuss a possible experimental realization of a sinusoidal vibration. Fig. 4.3 illustrates the spring-mass system consisting of a mass horizontally fixed to a spring. This system has a resting position, which we can set to the point $x = 0$, where no forces act on the mass. When elongated or compressed, the spring exerts a restoring force on the mass working to bring the mass back into its resting position,

$$F_{restore} = -kx \ . \tag{4.13}$$

This so-called *Hooke's law* holds for reasonably small elongations. The spring coefficient k is a characteristic of the spring.

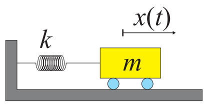


Figure 4.3: Illustration of the spring-mass system.

The oscillation frequency of the spring-mass system is determined by the spring coefficient and the mass, but the phase and the amplitude of the oscillation are parameters, that depend on the way the spring-mass is excited. Knowing the position and velocity of the oscillation at a given time, that is, the initial conditions of the motion, we can determine the amplitude and phase. To see this, we expand the general formula for a sinusoidal oscillation,

$$x(t) = A\cos(\omega_0 t - \phi) = A\cos(\omega_0 t)\cos\phi + A\sin(\omega_0 t)\sin\phi \tag{4.14}$$

and calculate the derivative,

$$v(t) = -A\omega_0\cos\phi\sin(\omega_0 t) + A\omega_0\sin\phi\cos(\omega_0 t) \ . \tag{4.15}$$

With the initial conditions $x(0) = x_0$ and $v(0) = v_0$ we get,

$$A\cos\phi = x_0 \qquad \text{and} \qquad A\omega_0\sin\phi = v_0 \ . \tag{4.16}$$

Hence,

$$x(t) = x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t) \ . \tag{4.17}$$

Solve the Excs. 4.1.10.3, 4.1.10.4, 4.1.10.5, and 4.1.10.6.

4.1.5 Energy conservation

Considerations of *energy conservation* can often help solving mechanical problems. The kinetic energy due to the movement of the mass m is,

$$E_{kin} = \tfrac{m}{2}v^2 \ , \tag{4.18}$$

and the potential energy due to the restoring force is,

$$E_{pot} = -\int_0^x F dx' = -\int_0^x -kx'dx' = \tfrac{k}{2}x^2 \ . \tag{4.19}$$

The total energy must be conserved:

$$E = E_{kin} + E_{pot} = \tfrac{m}{2}v^2 + \tfrac{k}{2}x^2 = const \ , \tag{4.20}$$

but is continuously transformed between kinetic energy and potential energy. This is illustrated on the left-hand side of the Fig. 4.4.

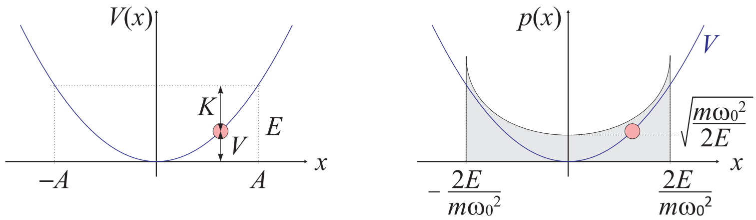


Figure 4.4: (Left) Energy conservation in the spring-mass system showing the kinetic energy K, the potential energy V, and the total energy E. (Right) Probability density of finding the oscillator in position x.

Example 9 (*Probability distribution in the harmonic oscillator*): Let us now use the principle of energy conservation to calculate the probability of finding the oscillating mass next to a given displacement x. For this, we solve the last equation by the velocity,

$$v = \frac{dx}{dt} = \sqrt{\frac{2}{m}E - \tfrac{k}{m}x^2} = \omega_0\sqrt{\frac{2E}{m\omega_0^2} - x^2} \ , \tag{4.21}$$

or

$$\frac{dx}{\sqrt{\frac{2E}{m\omega_0^2} - x^2}} = \omega_0 dt \ . \tag{4.22}$$

The probability of finding the mass within a given time interval dt is,

$$p(t)dt = \frac{dt}{T} = \frac{\omega_0}{2\pi}dt = \frac{dx}{2\pi\sqrt{\frac{2E}{m\omega_0^2} - x^2}} = \tilde{p}(x)dx \ . \tag{4.23}$$

Hence,

$$\tilde{p}(x) = \frac{1}{2\pi\sqrt{\frac{2E}{m\omega_0^2} - x^2}} \tag{4.24}$$

is the probability density of finding in the mass at the position $x(t)$. Using $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}$ with $x_0 = \sqrt{\frac{2E}{m\omega_0^2}}$ we verify,

$$2\int_{-x_0}^{x_0} \tilde{p}(x)dx = \frac{1}{\pi}\left[\arcsin\frac{x}{\sqrt{\frac{2E}{m\omega_0^2}}}\right]_{-x_0}^{x_0} = \frac{2}{\pi}\arcsin\frac{x_0}{\sqrt{\frac{2E}{m\omega_0^2}}} = \frac{2}{\pi}\arcsin 1 = 1 \ . \tag{4.25}$$

The probability density is shown on the right side of Fig. 4.4 [1].

[1] To understand the difference between the probability densities $p(t)$ and $\tilde{p}(x)$ we imagine the following experiments: We divide the period T into equal intervals dt and take a series of photos, all with the same exposure time dt. To understand the meaning of $p(t)$, we throw a random number to choose one of the photos. Each photo has the same probability dt/T to be chosen and, of course, $\int_0^T p(t)dt = 1$. To understand the meaning of $\tilde{p}(x)$, we identify the position of the oscillator in each photo and plot it in a histogram. This histogram is reproduced by $\tilde{p}(x)$.

4.1.6 The spring-mass system with gravity

When a mass is suspended vertically to a spring, as shown on the left-hand side of Fig. 4.5, the gravitational force acts on the mass in addition to the restoring force. This can be expressed by the following balance of forces,

$$ma = -ky - mg \ , \tag{4.26}$$

letting the y-axis be positive in the direction opposite to gravitation. Replacing $\tilde{y}' \equiv y - y_0$ with $y_0 \equiv -\frac{mg}{k}$, we obtain,

$$m\tilde{a} = -k\tilde{y} \ . \tag{4.27}$$

Therefore, the movement is the same as in the absence of gravitation, but around an equilibrium point shifted downward by y_0.

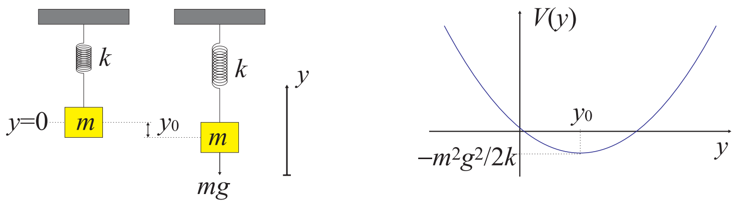


Figure 4.5: Left: Vertical spring-mass system. Right: Conservation of energy in the spring-mass system with gravity.

Energy conservation is now generalized to,

$$E = E_{kin} + E_{mol} + E_{grv} = \frac{m}{2}v^2 + \frac{k}{2}y^2 + mgy = const \ , \tag{4.28}$$

the potential energy being,

$$\begin{aligned} E_{pot} = E_{mol} + E_{grv} &= \tfrac{k}{2}y^2 + mgy \qquad (4.29) \\ &= \tfrac{k}{2}(y-y_0)^2 + \tfrac{k}{2}2y_0y - \tfrac{k}{2}y_0^2 + mgy = \tfrac{k}{2}(y-y_0)^2 - \frac{m^2g^2}{2k} \ . \end{aligned}$$

The right-hand side of Fig. 4.5 illustrates the conservation of energy in the spring-mass system with gravity. See Excs. 4.1.10.7 to 4.1.10.9, 4.1.10.10, and 4.1.10.11.

4.1.7 The pendulum

The pendulum is another system which oscillates in the gravitational field. In the following, we will distinguish three different types of pendulums. In the *ideal pendulum* the mass of the oscillating body is all concentrated in one point and the oscillations have small amplitudes. In the *physical pendulum* the mass of the body is distributed over a finite spatial region. And *mathematical pendulum* is a point mass oscillating with a large amplitude and therefore subject to a nonlinear restoring force.

Figure 4.6: Physical pendulum.

4.1.7.1 The ideal pendulum

The *ideal pendulum* is schematized on the left side of Fig. 4.6. As the centrifugal force is compensated for by the traction of the wire supporting the mass, the acceleration force ma is solely due to the perpendicular projection $-mg\sin\theta$ on the wire. For small amplitudes, $\sin\theta \simeq \theta$, such that [2],

$$ma \simeq -mg\theta \ . \tag{4.30}$$

The tangential acceleration is now,

$$a = \dot{v} = \ddot{s} = \frac{d}{dt}\theta L = L\ddot{\theta} \ . \tag{4.31}$$

Thus,

$$\ddot{\theta} + \frac{g}{L}\theta \simeq 0 \ . \tag{4.32}$$

This equation has the same structure as that of the already studied spring-mass system $\ddot{x} + \frac{k}{m}x = 0$. Therefore, we can deduce that the ideal pendulum oscillates with the frequency,

$$\omega_0 = \sqrt{\frac{g}{L}} \ , \tag{4.33}$$

only that the oscillating degree of freedom is an angle rather than a spatial shift. It is interesting to note that the oscillation frequency is independent of the mass. See Exc. 4.1.10.12.

4.1.7.2 The physical pendulum

We consider an irregular body suspended at a point P as schematized on the right-hand side of Fig. 4.6. The center-of-mass be displaced from the suspension point by a distance D. This system represents the *physical pendulum*. Gravitation exerts a torque $\vec{\tau}$ on the center-of-mass,

$$\vec{\tau} = \mathbf{D} \times m\mathbf{g} \qquad \text{with} \qquad \tau = I\ddot{\theta} \ , \tag{4.34}$$

where I is the moment of inertia of the body for rotations about the suspension axis. Like this,

$$I\ddot{\theta} = -Dmg\sin\theta \ . \tag{4.35}$$

[2]The equation of motion can be derived from the Hamiltonian $H = \frac{L_\theta^2}{2ml^2} + mgl\cos\theta$ using $\dot{\theta} = \partial H/\partial L_\theta$ and $\dot{L}_\theta = -\partial H/\partial\theta$, where L_θ is the angular momentum.

Considering once more small angles, $\sin\theta \simeq \theta$, we obtain,

$$\ddot{\theta} + \omega_0^2\theta \simeq 0 \qquad \text{with} \qquad \omega_0 \equiv \sqrt{\frac{Dmg}{I}} \ . \tag{4.36}$$

It is worth mentioning that the inertial moment of a body whose mass is concentrated in a point at a distance D from the suspension point follows *Steiner's law*,

$$I = mD^2 \ . \tag{4.37}$$

With this we recover the expression of the ideal pendulum,

$$\omega_0 = \sqrt{\frac{Dmg}{mD^2}} = \sqrt{\frac{g}{D}} \ . \tag{4.38}$$

4.1.7.3 The mathematical pendulum

The equation describing the mathematical pendulum (see Fig. 4.6) has already been derived but, differently from what we did before, here we will not apply the small angle approximation,

$$\ddot{\theta} = -\frac{g}{L}\sin\theta = -\omega_0^2\sin\theta \ . \tag{4.39}$$

Energy conservation can be formulated as follows:

$$\begin{aligned} 0 &= \frac{dE}{dt} = \frac{d}{dt}(E_{rot} + E_{pot}) = \frac{d}{dt}\frac{I}{2}\dot{\theta}^2 + \frac{d}{dt}mgL(1-\cos\theta) \\ &= \frac{I}{2}2\dot{\theta}\ddot{\theta} + mgL\dot{\theta}\sin\theta \simeq \dot{\theta}(I\ddot{\theta} + mgL\theta) \ . \end{aligned} \tag{4.40}$$

Thus, we obtain the same differential equation,

$$\ddot{\theta} + \frac{mgL}{I}\theta = 0 \ . \tag{4.41}$$

Example 10 (*Simulation of an anharmonic pendulum*): When the anharmonicity is not negligible, it is impossible to solve the differential equation analytically. We must resort to numerical simulations. The simplest procedure is an iteration of the type,

$$\begin{aligned} \theta(t+dt) &= \theta(t) + dt\dot{\theta} = \theta(t) + dt\omega \\ \omega(t+dt) &= \omega(t) + dt\dot{\omega} = \omega(t) - dt\omega_0\sin\theta \ . \end{aligned}$$

Fig. 4.7(a) shows the temporal dephasing of the oscillation caused by the anharmonicity as compared to the harmonic oscillation. Fig. 4.7(b) shows the orbits $\theta(t) \longmapsto \omega(t)$ in the phase space.

Do the Excs. 4.1.10.17 to 4.1.10.18.

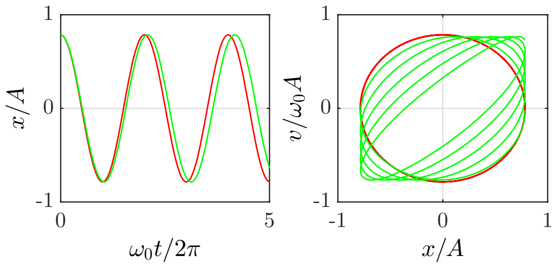


Figure 4.7: (code) Diffusion due to anharmonicities (a) in time and (b) in phase space. The red curves show the harmonic approximation.

4.1.8 The spring-cylinder system

Another example of an oscillating system is shown in Fig. 4.8. The inertial moment of the cylinder is $I = \frac{M}{2}R^2$. The spring exerts the force,

$$F_{mol} = -kx \ . \tag{4.42}$$

Therefore, we have the equations of motion,

$$\begin{aligned} M\ddot{x} &= F_{mol} - F_{at} \\ I\ddot{\theta} &= -RF_{at} \ . \end{aligned} \tag{4.43}$$

If the wheel does not slip, we can eliminate the friction using $x = R\omega$, and we obtain,

$$I\ddot{\theta} = I\frac{\ddot{x}}{R} = \frac{M}{2}R^2\frac{\ddot{x}}{R} = -RF_{at} = -R(-kx - M\ddot{x}) \ . \tag{4.44}$$

Resolving by $\ddot{x}$,

$$\ddot{x} + \frac{2k}{3M}x = 0 \ . \tag{4.45}$$

The frequency is,

$$\omega_0 = \sqrt{\frac{2k}{3M}} \ . \tag{4.46}$$

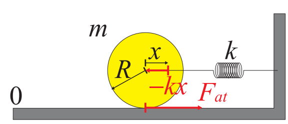


Figure 4.8: The spring-cylinder system.

4.1.9 Two-body oscillation

We now consider the oscillations of two bodies m_1 and m_2 located at the positions x_1 and x_2 and interconnected by a spring k, as shown in Fig. 4.9. The free length, that is, the distance at which the spring exerts no forces on the masses, is ℓ. The forces grow with the stretch $x \equiv x_2 - x_1 - \ell$ of the spring, such that $x > 0$ when the spring is stretched and $x < 0$ when it is compressed. Thereby,

$$m_1\ddot{x}_1 = kx \qquad \text{and} \qquad m_2\ddot{x}_2 = -kx \; . \tag{4.47}$$

Adding these equations,

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 \equiv (m_1 + m_2)\ddot{x}_{cm} = 0 \; . \tag{4.48}$$

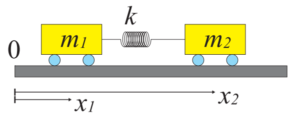


Figure 4.9: Two bodies in relative vibration.

Dividing the equations by the masses and subtracting them,

$$\ddot{x}_1 - \ddot{x}_2 = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)x = \ddot{x}_{rel} = -\frac{k}{\mu}x = \omega_0 x \; , \tag{4.49}$$

where $\omega_0^2 = k/\mu$ and $\mu^{-1} \equiv m_1^{-1} + m_2^{-1}$ is called the *reduced mass*. The introduction of the reduced mass turns the oscillator consisting of two bodies equivalent to a system consisting of only one mass and one spring, but with an increased vibration frequency,

$$\omega_\mu = \sqrt{\frac{k}{\mu}} = \sqrt{2\frac{k}{m}} \; . \tag{4.50}$$

This system represents an important model for the description of *molecular vibration*. Note that for $m_1 \longrightarrow \infty$ we restore the known situation of a spring-mass system fixed to a wall.

4.1.10 Exercises

4.1.10.1 Ex: Zenith in São Carlos

Knowing that the latitude of the Sun in the tropics of Capricorn is $\alpha_{trop} = 23°$ calculate at what time of the year the sun is vertical at noon in São Carlos, SP, Brazil.

4.1.10.2 Ex: Length of days on Earth

Calculate the length of a day on Earth as a function of the location's longitude ϕ and latitude θ and of the season of the year.

4.1.10.3 Ex: Swing modes

In the systems shown in the figure there is no friction between the surfaces of the bodies and floor, and the springs have negligible mass. Find the natural oscillation frequencies.

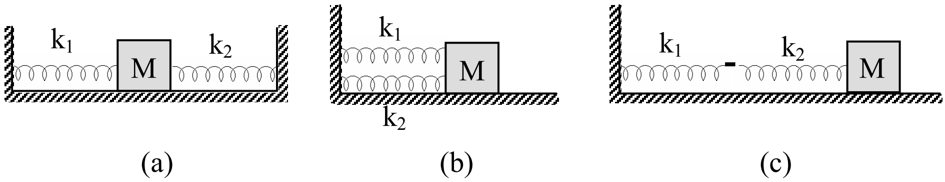


Figure 4.10: Swing modes.

4.1.10.4 Ex: Coupled springs

A mass m is suspended within a horizontal ring of radius $R = 1\,\mathrm{m}$ by three springs with the constants $D_1 = 0.1\,\mathrm{kg/m}$, $D_2 = 0.2\,\mathrm{N/m}$, and $D_3 = 0.3\,\mathrm{N/m}$. The suspension points of the springs on the ring have the same mutual distances. Determine the equilibrium position of the mass assuming that the springs' extensions at rest range is 0.

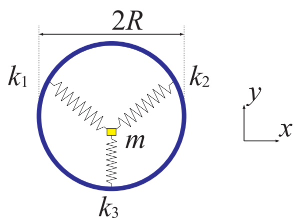


Figure 4.11: Coupled springs.

4.1.10.5 Ex: Coupled springs

A mass m is suspended by four springs with the constants k_n, as shown in the figure. Determine the equilibrium position of the mass. Assume the ideal case of ideally compressible springs.

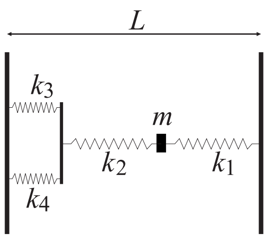


Figure 4.12: Coupled springs.

4.1.10.6 Ex: Coupled springs

Calculate the resulting spring constants for the constructions shown in the scheme. Individual springs are arbitrarily compressible with spring constants D_k.

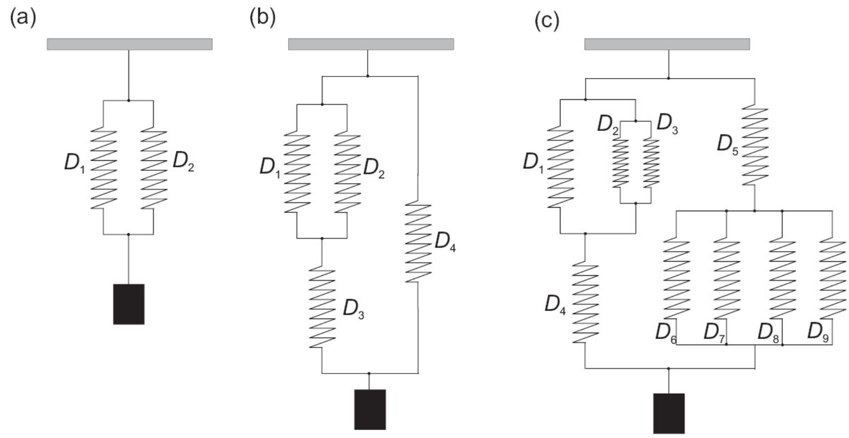


Figure 4.13: Coupled springs.

4.1.10.7 Ex: Spring-mass system

A body of unknown mass hangs at the end of a spring, which is neither stretched nor compressed, and is released from rest at a certain moment. The body drops a distance y_1 until it rests for the first time after the release. Calculate the period of oscillatory motion.

4.1.10.8 Ex: Spring-mass system

A body of $m = 1.5\,\mathrm{kg}$ stretches a spring by $y_0 = 2.8\,\mathrm{cm}$ from its natural length when being at rest. Now, we let it swing at this spring with an amplitude of $y_m = 2.2\,\mathrm{cm}$.
a. Calculate total energy of the system.
b. Calculate the gravitational potential energy at the body's lower turning point.
c. Calculate the potential energy of the spring at the body's lower turning point.
d. What is the maximum kinetic energy of the body (when $U = 0$ is the point where the spring is at equilibrium).

4.1.10.9 Ex: Energy conservation in the harmonic oscillator

Consider a mass of $m = 0.1\,\mathrm{kg}$ that vibrates on a spring with the spring constant $C = 10\,\mathrm{N/m}$. The maximum deflection A around the rest position of the spring is $10\,\mathrm{cm}$.
a. At what frequency does the mass vibrate?
b. Give the potential and the kinetic energy of the mass as a function of the displacement $x(t)$ resp. $\dot{x}(t)$.
c. Derive the equation of motion for the deflection from the energy theorem. The initial condition is chosen so that at time $t = 0$ the mass is at the zero crossing ($x(0) = 0$) and has maximum speed. **First help:** Integrate using the *variable separation* method. **Second help:** $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a}$.

4.1.10.10 Ex: U-shaped water tube

Consider a U-shaped tube filled with water. The total length of the water column is L. Exerting pressure on one tube outlet the column is incited to perform oscillations. Calculate the period of the oscillation.

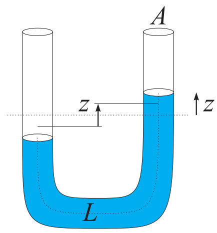


Figure 4.14: U-shaped water tube.

4.1.10.11 Ex: Buoy in the sea

A hollow cylindrical buoy with cross-sectional area A and mass M floats in the sea so that the axis of symmetry is aligned with gravitation. An albatross of mass m sitting on the buoy waits until time $t = 0$ and takes off. With which frequency and amplitude does the buoy oscillate if friction can be neglected? Derive the equation of motion and the complete solution.

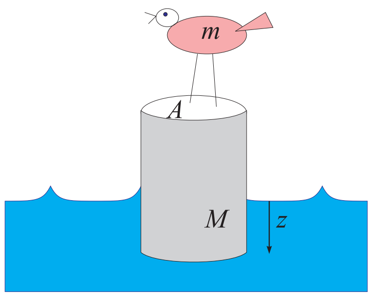


Figure 4.15: Fluctuating buoy.

4.1.10.12 Ex: Complicated pendulum oscillation

At a distance of $d = 30\,\text{cm}$ below the suspension point of a pendulum with the length $l_1 = 50\,\text{cm}$ there is a fixed pin S on which the wire suspending the pendulum temporarily bends during vibration. How many vibrations does the pendulum perform per minute?

4.1.10.13 Ex: Physical pendulum

Calculate the oscillation frequency of a thin bar of mass m and length L suspended at one end.

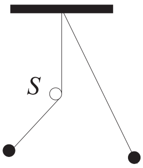


Figure 4.16: Mathematical pendulum.

4.1.10.14 Ex: Physical pendulum

An irregularly shaped flat body has the mass $m = 3.2\,\text{kg}$ and is hung on a massless rod with adjustable length, which is free to swing in the plane of the body itself. When the rod's length is $L_1 = 1.0\,\text{m}$, the period of the pendulum is $t_1 = 2.6\,\text{s}$. When the rod is shortened to $L_2 = 0.8\,\text{m}$, the period decreases to $t_2 = 2.5\,\text{s}$. What is the period of the oscillation when the length is $L_3 = 0.5\,\text{m}$?

4.1.10.15 Ex: Physical pendulum

A physical pendulum of mass M consists of a homogeneous cube with the edge length d. As shown in the figure, the pendulum is hung without friction on a horizontal rotation axis.
a. Determine the inertial momentum about the rotation axis using Steiner's theorem.
b. The pendulum now performs small oscillations around its resting position. Determine the angular momentum.
c. Give the equation of motion for small pendulum amplitudes ϕ around its resting position and the oscillation period.

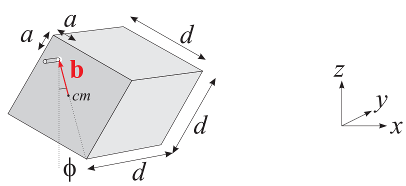


Figure 4.17: Physical pendulum.

4.1.10.16 Ex: Physical pendulum on a spiral spring

Consider a beam of mass $m = 1\,\text{kg}$ with the dimensions $(a, b, c) = (3\,\text{cm}, 3\,\text{cm}, 8\,\text{cm})$. The beam is rotatable about an axis through the point A. At point B, at a distance r from point A, the beam is fixed to a spiral spring exerting the retroactive force $\mathbf{F}_R = D\vec{\phi}$ with $D = 100\,\text{N/m}$. Determine the differential equation of motion and solve it. Determine the period of the oscillation.

4.1.10.17 Ex: Anharmonic potential

A mass m moves in a potential $V(x)$, which has a minimum at $V(0) = 0$ for $x = 0$ and a first maximum at x_M. At time $t = 0$ the mass is at point $x = 0$. The initial speed

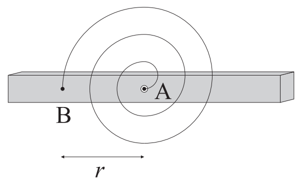


Figure 4.18: Physical pendulum on a spiral spring.

is chosen so that, when it reaches the maximum of the potential at x_M its velocity is zero. Calculate for the two cases $V(x) = \alpha x^2 - \beta x^4$ and $V(x) = \lambda(1 - \cos^2 \kappa x)$:
a. the coordinates of the maximum x_M and $V(x_M)$;
b. with the help of the energy conservation law the total energy E;
c. the implicit equation of motion $t = g(x)$ or, if possible, the explicit one $x = f(t)$; How long does the mass take to reach the point x_M. Justify the result by physical arguments.

4.1.10.18 Ex: Anharmonic potential

A particle with the mass $m = 1.4 \times 10^{-25}$ kg moves along the positive x-axis. A constant force directed towards the origin of the coordinate system of $B = 10^{-23}$ N and a repulsive force of A/x^2 act on the particle, whereby $A = 10^{-35}$ Nm^2.
a. Calculate the potential energy function $V(x)$.
b. Sketch the energy as a function of the location x when the maximum kinetic energy is $T_0 = 10^{-28}$ J.
c. Determine the equilibrium position x_0 (force $F(x_0) = 0$) and the turning points (velocity $\dot{x} = 0$).
d. How large is the frequency of small vibrations around x_0?

4.1.10.19 Ex: Accelerated pendulum

A simple pendulum of length L is attached to a cart that slides without friction downward an plane inclined by an angle α with respect to the horizontal. Determine the oscillation period of the pendulum on the cart.

4.1.10.20 Ex: Accelerated pendulum

a. A pendulum of length L and mass M is suspended from the roof of a wagon horizontally accelerated with the acceleration a_{ext}. Find the equilibrium position of the pendulum. Determine the oscillation frequency for small oscillations and derive the differential equation of motion for an observer sitting in the wagon. (Note that you cannot assume small displacements, if the acceleration a_{ext} is large.)
b. In the same wagon there is a mass m connected to the front wall by a spring k. Find the equilibrium position of the mass. Determine the oscillation frequency and derive the differential motion equation for an observer sitting in the wagon.

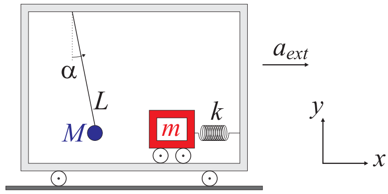


Figure 4.19: Accelerated pendulum.

4.1.10.21 Ex: Oscillation of a rolling cylinder

Consider a cylinder secured by two springs that rotates without sliding, as shown in the figure. Calculate the frequency for small oscillations of the system.

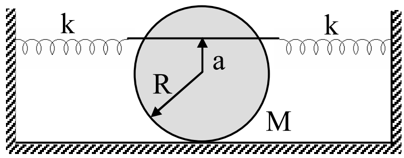


Figure 4.20: Rolling cylinder.

4.1.10.22 Ex: Rocking chair

Consider a thin rod of mass M and length $2L$ leaning on its center-of-mass, as shown in the figure. It is attached at both ends by springs of constants k. Calculate the angular frequency for small oscillations of the system.

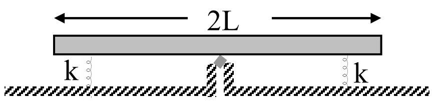


Figure 4.21: Rocking chair.

4.1.10.23 Ex: Rotational oscillation of a disk

Consider a disk of mass M and radius R ($I = \frac{1}{2}MR^2$) that can rotate around the polar axis. A body of mass $m$hangs at an ideal rope that runs through the disk (without slipping) and is attached to a wall by a spring of constant k, as shown in the figure. Calculate the natural oscillation frequency of the system.

4.1.10.24 Ex: Oscillation of a half cylinder

Consider a massive, homogeneous half-cylinder of mass M and radius R resting on a horizontal surface. If one side of this solid is slightly pushed down and released, it will swing around its equilibrium position. Determine the period of this oscillation.

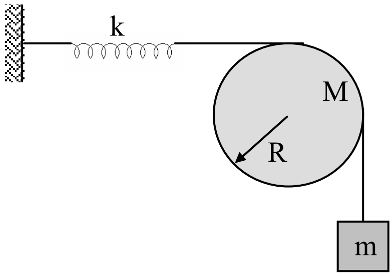


Figure 4.22: Rotational oscillation of a disk.

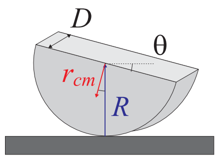


Figure 4.23: Oscillation of a half cylinder.

4.1.10.25 Ex: Pendulum coupled to a spring

Consider a simple pendulum of mass m and length L, connected to a spring of constant k, as shown in the figure. Calculate the frequency of the system for small oscillation amplitudes.

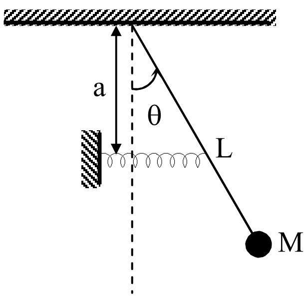


Figure 4.24: Pendulum coupled to a spring.

4.1.10.26 Ex: Pendulum carousel

A mass m is hung by a rope of length l on a carousel with the radius R. The pendulum performs small amplitude oscillations in the direction of the rotation axis of the carousel. How does the period of oscillation depend on the rotation speed of the carousel?

4.2 Superposition of periodic movements

Several movements that we already know can be understood as superpositions of periodic movements in different directions and, possibly, with different phases. Example are the circular or elliptical motion of a planet around the sun or the Lissajous figures. In these cases, the motion must be described by vectors, $\mathbf{r}(t) \equiv (x(t), y(t))$. It is also possible to imagine superpositions of periodic movements in the same degree of freedom. The movement of the membrane of a loudspeaker or musical instruments usually vibrates harmonically, but follows a *superposition* of harmonic oscillations. According to the *superposition principle*, we will take the resultant of several harmonic vibrations as the sum of the individual vibrations.

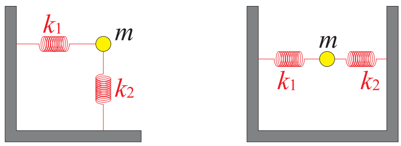


Figure 4.25: Superposition of vibrations in different (left) and equal (right) degrees of freedom.

4.2.1 Rotations and complex notation

We now consider a uniform circular motion. The radius of the circle being R, the motion is completely described by the angle $\theta(t)$ which grows uniformly,

$$\theta = \omega t + \alpha \ . \tag{4.51}$$

The projections of the movement in x and y are,

$$x(t) = A\cos\theta \qquad \text{and} \qquad y(t) = A\sin\theta \ . \tag{4.52}$$

Thus, we can affirm $x(t) = y(t + \pi/2)$, that is, the projections have a mutual phase shift of $\pi/2$.

The circular motion can be represented in the complex plane using the imaginary unit $i \equiv \sqrt{-1}$ and Euler's relationship $e^{\imath\theta} = \cos\theta + \imath\sin\theta$, as illustrated in Fig. 4.26.[3,4] With $r = Ae^{\imath\theta}$ we obtain $x = A\mathfrak{Re}\, e^{\imath\theta}$ and $\imath y = A\mathfrak{Im}\, e^{\imath\theta}$ and $r = x + \imath y$.

We will use the complex notation extensively, as it greatly facilitates the calculation.

4.2.2 Lissajous figures

Other periodic movements in the two-dimensional plane are possible, when the movements in x and y have different phases or frequencies. These are called *Lissajous figures*.

[3]The Euler relation can easily be derived by Taylor expansion.

[4]To check your notions on complex numbers do the exercises in Chp. 1 of the Book of A.P. French.

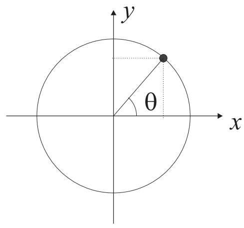


Figure 4.26: Circular motion in the complex plane.

We consider a body subject to two harmonic movements in orthogonal directions:

$$x(t) = A_x \cos(\omega_x t + \varphi_x) \qquad \text{and} \qquad y(t) = A_y \cos(\omega_y t + \varphi_y) \ . \tag{4.53}$$

When ω_x/ω_y is a rational number, the curve is closed and the motion repeats after equal time periods. The upper charts in Fig. 4.27 show trajectories of the body for $\omega_x/\omega_y = 1/2$, $1/3$, and $2/3$, letting $A_x = A_y$ and $\varphi_x = \varphi_y$. The lower charts in Fig. 4.27 show trajectories for $\omega_x/\omega_y = 1/2$, $1/3$, letting $\varphi_x - \varphi_y = 0$, $\pi/4$, and $\pi/2$.

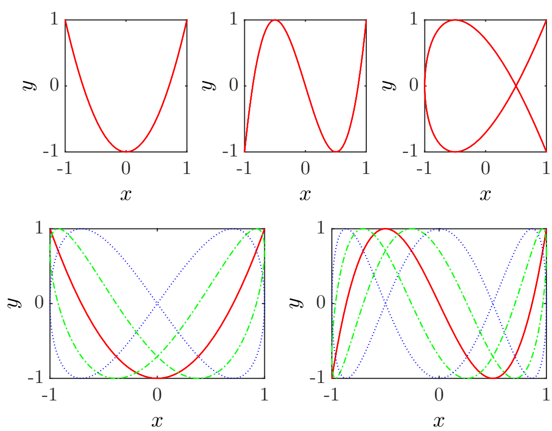


Figure 4.27: (code) Trajectories of a body oscillating with different frequencies in two dimensions.

Example 11 (*Lissajous figures*):

- Connect two function generators to the two channels of an oscilloscope in x-y.
- MATLAB simulation.

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4.2.3 Vibrations with equal frequencies superposed in one dimension

Vibratory movements can overlap. The result can be described as a sum,

$$\begin{aligned} x(t) &= x_1(t) + x_2(t) = A_1\cos(\omega t+\alpha_1) + A_2\cos(\omega t+\alpha_2) \\ &= \text{Re}[A_1e^{\imath(\omega t+\alpha_1)} + A_2e^{\imath(\omega t+\alpha_2)}] = \text{Re}e^{\imath\omega t}[A_1e^{\imath\alpha_1} + A_2e^{\imath\alpha_2}] \ . \end{aligned} \tag{4.54}$$

That is, the new motion is a cosine vibration, $x(t) = A\cos\omega t$, with the phase,

$$\tan\alpha = \frac{\text{Im } x(0)}{\text{Re } x(0)} = \frac{\text{Im}(A_1e^{\imath\alpha_1} + A_2e^{\imath\alpha_2})}{\text{Re}(A_1e^{\imath\alpha_1} + A_2e^{\imath\alpha_2})} = \frac{A_1\sin\alpha_1 + A_2\sin\alpha_2}{A_1\cos\alpha_1 + A_2\cos\alpha_2} \ , \tag{4.55}$$

and the amplitude,

$$A = |A_1e^{\imath\alpha_1} + A_2e^{\imath\alpha_2}| = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\alpha_1-\alpha_2)} \ . \tag{4.56}$$

We consider the case $A_1 = A_2$,

$$\tan\alpha = \frac{\sin\alpha_1 + \sin\alpha_2}{\cos\alpha_1 + \cos\alpha_2} \quad , \quad A = 2A\cos\frac{\alpha_1-\alpha_2}{2} \ . \tag{4.57}$$

The cases $\alpha_1 = \alpha_2$ or $\alpha_2 = 0$ further simplify the result.

4.2.4 Frequency beat

Vibratory movements with different frequencies can overlap. The result can be described as a sum,

$$x(t) = x_1(t) + x_2(t) = A_1\cos\omega_1 t + A_2\cos\omega_2 t = \mathfrak{Re}\left[A_1e^{\imath\omega_1 t} + A_2e^{\imath\omega_2 t}\right] \ . \tag{4.58}$$

Considering the case $A_1 = A_2$ we obtain,

$$\begin{aligned} x(t) &= A\mathfrak{Re}\left[e^{\imath\omega_1 t} + e^{\imath\omega_2 t}\right] \\ &= A\mathfrak{Re}\left[e^{\imath(\omega_1+\omega_2)t/2}e^{\imath(\omega_1-\omega_2)t/2} + e^{\imath(\omega_1+\omega_2)t/2}e^{-\imath(\omega_1-\omega_2)t/2}\right] \\ &= A\mathfrak{Re}\, e^{\imath(\omega_1+\omega_2)t/2} 2\cos\frac{(\omega_1-\omega_2)t}{2} \\ &= 2A\cos\frac{(\omega_1+\omega_2)t}{2}\cos\frac{(\omega_1-\omega_2)t}{2} \ . \end{aligned} \tag{4.59}$$

Example 12 (*Amplitude modulation*): An important example is the amplitude modulation of radiofrequency signals.

Example 13 (*Visualization of beat frequencies on an oscilloscope*):

- Connect two function generators to the two channels of an oscilloscope and add the channels.
- MATLAB simulation.
- Modulate one signal by another in a frequency mixer.

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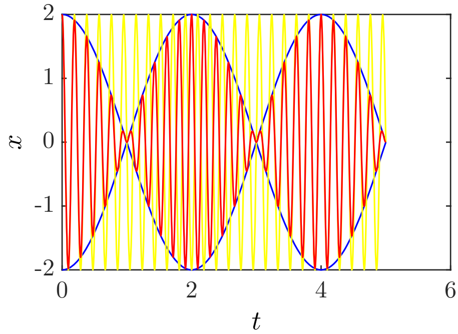

Figure 4.28: (code) Illustration of the beating of two frequencies $\nu_1 = 5.5\,\text{Hz}$ and $\nu_2 = 5\,\text{Hz}$ showing the perceived vibration (red), the vibration with the frequency $(\nu_1 + \nu_2)/2$ (blue), and the vibration with frequency $(\nu_1 - \nu_2)/2$ (yellow).

4.2.5 Amplitude and frequency modulation

Radio frequencies above 300 kHz can easily be emitted and received by antennas, while audio frequencies are below 20 kHz. However, radio frequencies can be used as carriers for audio frequencies. This can be done by modulating the audio signal on the *amplitude* of the carrier () before sending the carrier frequency. The receiver retrieves the audio signal by demodulating the carrier. Therefore, audio signals can be transmitted by electromagnetic waves. Another technique consists in modulating the *frequency* of these waves (). We will now calculate the spectrum of these two modulations using complex notation and show how to demodulate the encoded audio signals by multiplication with a local oscillator corresponding to the carrier wave.

4.2.5.1 AM

Let ω and Ω be the frequencies of the carrier wave and the modulation, respectively. We can describe the amplitude modulation by,

$$U(t) = (1 + S(t)) \cos \omega t \; . \tag{4.60}$$

After the receiver has registered this signal, we demodulate it by multiplying it with $\cos \omega t$:

$$U(t) \cos \omega t = (1 + S(t)) \cos^2 \omega t = (1 + S(t)) \left(\tfrac{1}{2} + \tfrac{1}{2} \cos 2\omega t \right) \; . \tag{4.61}$$

We purify this signal passing it through a low-pass filter eliminating the rapid oscillations:

$$U(t) \cos \omega t \longrightarrow \tfrac{1}{2}(1 + S(t)) \; . \tag{4.62}$$

We retrieve the original signal $S(t)$.

4.2.5.2 FM

We can describe the frequency modulation by,

$$U(t) = e^{\imath(\omega t + N \sin \Omega t)} = e^{\imath \omega t} \sum_{k=-\infty}^{\infty} \mathcal{J}_k(N) e^{\imath k \Omega t} \; . \tag{4.63}$$

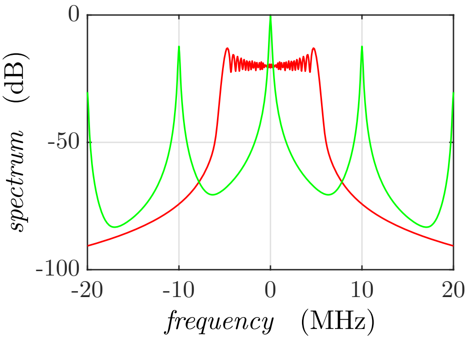

Figure 4.29: (code) Modulation signal.

The modulation of the carrier wave generates sidebands. This can be seen by expanding the signal carrying the phase modulation into a Fourier series,

$$e^{\imath\omega t}\sum_{k=-\infty}^{\infty}\mathcal{J}_k(\beta)e^{\imath k\Omega t}\simeq e^{\imath\omega t}+\mathcal{J}_1(N)e^{\imath\omega t+\imath\Omega t}+\mathcal{J}_{-1}(N)e^{\imath\omega t-\imath\Omega t} \tag{4.64}$$

when the *modulation index* N is small. Here, $J_{-k}(N)=(-1)^kJ_k(N)$ are the Bessel functions.

The spectrum of a signal with PM modulation consists of discrete lines, called sidebands, whose amplitudes are given by Bessel functions,

$$S(\omega)=\sum_{k=-\infty}^{\infty}|\mathcal{J}_k(N)|^2\delta(\omega+k\Omega)\ . \tag{4.65}$$

In real systems, the frequency bands have finite widths β due to frequency noise or to the finite resolution of the detectors,

$$S(\omega)=\sum_{k=-\infty}^{\infty}|\mathcal{J}_k(N)|^2\frac{N^2}{(\omega-k\Omega)^2+N^2}\ . \tag{4.66}$$

Example 14 (*Frequency spectrum*):

- Modulate the frequency of a VCO.
- Show in the spectrum analyzer the transition to sidebands.

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4.2.6 Exercises

4.2.6.1 Ex: Amplitude modulation

Consider a carrier wave of $\omega/2\pi=1$ MHz frequency whose amplitude is modulated by an acoustic signal of $\Omega/2\pi=1\,\text{kHz}$: $U(t)=A\cos\Omega t\cos\omega t$. To demodulate the signal, multiply the received wave $U(t)$ by the carrier radiofrequency. Interpret the result.

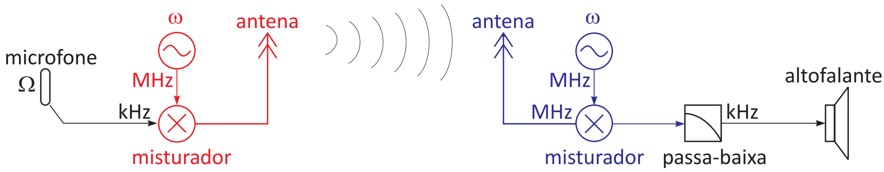

Figure 4.30: Illustration of radiofrequency signal transmission.

4.3 Damped and forced vibrations

Frequently, vibrations are exposed to external perturbations. For example, damping forces due to friction exerted by the medium in which vibration takes place work to waste and dissipate the energy of the oscillation and, therefore, to reduce the amplitude of the oscillation. In contrast, periodic forces can pump energy into the oscillator system and excite vibrations.

4.3.1 Damped vibration and friction

Let us first deal with damping by forces named *Stokes friction*, that is, forces which are proportional to the velocity of the oscillating mass and contrary to the direction of motion, $F_{frc} = -bv$, where b is the friction coefficient. With this additional term, the equation of motion is,

$$ma = -bv - kx \ . \tag{4.67}$$

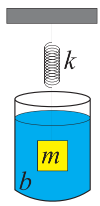

Figure 4.31: Oscillation damped by a viscous medium.

The calculation of the *damped oscillator* can be greatly simplified by the use of complex numbers by making the ansatz,

$$x(t) = Ae^{\lambda t} \ , \tag{4.68}$$

where λ is a complex number. We get,

$$m\lambda^2 + b\lambda + k = 0 \ , \tag{4.69}$$

giving the characteristic equation,

$$\lambda = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \ , \tag{4.70}$$

with

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \text{and} \qquad \gamma = \frac{b}{2m} \ . \tag{4.71}$$

The friction determines the damping behavior. We distinguish three cases discussed in the following sections.

4.3.1.1 Overdamped case

In the overdamped case, for $\omega_0 < \gamma$, there are two real solutions $\lambda = -\gamma \pm \kappa$ with $\kappa \equiv \sqrt{\gamma^2 - \omega_0^2}$ for the characteristic equation, giving,

$$x(t) = e^{-\gamma t}(Ae^{-\kappa t} + Be^{\kappa t}) \ . \tag{4.72}$$

Choosing the initial conditions,

$$x_0 = x(0) = e^{-\gamma t}(Ae^{-\kappa t} + Be^{\kappa t}) = A + B \tag{4.73}$$
$$0 = v(0) = -A(\gamma+\kappa)e^{-(\gamma+\kappa)t} - B(\gamma-\kappa)e^{-(\gamma-\kappa)t} = -A(\gamma+\kappa) - B(\gamma-\kappa) \ ,$$

we determine the amplitudes,

$$A = \frac{x_0}{2}\left(1 - \frac{\gamma}{\kappa}\right) \qquad \text{and} \qquad B = \frac{x_0}{2}\left(1 + \frac{\gamma}{\kappa}\right) \ . \tag{4.74}$$

Finally, the solution is [5],

$$x(t) = x_0 e^{-\gamma t}\left[\cosh\kappa t + \frac{\gamma}{\kappa}\sinh\kappa t\right] \ . \tag{4.75}$$

4.3.1.2 Underdamped case

In the underdamped case, for $\omega_0 > \gamma$, we have two complex solutions $\lambda = -\gamma \pm \imath\omega$ with $\omega \equiv \sqrt{\omega_0^2 - \gamma^2}$, giving,

$$x(t) = e^{-\gamma t}(Ae^{\imath\omega t} + Be^{-\imath\omega t}) \ . \tag{4.76}$$

Choosing the initial conditions,

$$x_0 = x(0) = e^{-\gamma t}(Ae^{\imath\omega t} + Be^{-\imath\omega t}) = A + B \tag{4.77}$$
$$0 = v(0) = -A(\gamma-\imath\omega)e^{-(\gamma-\imath\omega)t} - B(\gamma+\imath\omega)e^{-(\gamma+\imath\omega)t} = -A(\gamma-\imath\omega) - B(\gamma+\imath\omega) \ ,$$

we determine the amplitudes,

$$A = \frac{x_0}{2}\left(1 + \frac{\gamma}{\imath\omega}\right) \qquad \text{and} \qquad B = \frac{x_0}{2}\left(1 - \frac{\gamma}{\imath\omega}\right) \ . \tag{4.78}$$

Finally, the solution is [6],

$$x(t) = x_0 e^{-\gamma t}\left[\cos\omega t + \frac{\gamma}{\omega}\sin\omega t\right] \ . \tag{4.79}$$

---

[5] Note that for super-strong damping, we have $\kappa \simeq \gamma$ and therefore,

$$x(t) = Ae^{-2\gamma t} + B \ .$$

, This is nothing more than the solution of the equation of motion without restoring force, $ma = -bv$.

[6] Note that, for very weak damping, we have $\gamma \simeq 0$ and $\omega \simeq \omega_0$ and hence,

$$x(t) = Ae^{\imath\omega_0 t} + Be^{-\imath\omega_0 t} \ .$$

This is nothing more than the solution of the frictionless equation of motion, $ma = -kx$.

4.3.1.3 Critically damped case

In the critically damped case, for $\omega_0 = \gamma$, there is only one solution $\lambda = -\gamma$, giving

$$x(t) = Ae^{-\gamma t} \; . \tag{4.80}$$

Since one solution is not sufficient to solve a second order differential equation, we need to look for another linearly independent solution. We can try another ansatz,

$$x(t) = Bte^{\lambda t} \; , \tag{4.81}$$

resulting in the characteristic equation,

$$m(\lambda^2 te^{\lambda t} + 2\lambda e^{\lambda t}) + b(\lambda te^{\lambda t} + e^{\lambda t}) + kte^{\lambda t} = 0 \; . \tag{4.82}$$

The terms in $e^{\lambda t}$ and $te^{\lambda t}$ should disappear separately, giving,

$$2m\lambda + b = 0 \quad \text{and} \quad m\lambda^2 t + b\lambda t + kt = 0 \quad \Longrightarrow \quad \lambda = -\frac{b}{2m} = -\gamma = -\omega_0 \; . \tag{4.83}$$

Finally, the solution is,

$$x(t) = (A + Bt)e^{-\gamma t} \; . \tag{4.84}$$

Choosing the initial conditions,

$$\begin{aligned} x_0 = x(0) &= (A + Bt)e^{-\gamma t} = A \\ 0 = v(0) &= (-\gamma A - \gamma Bt + B)e^{-\gamma t} = -\gamma A + B \; , \end{aligned} \tag{4.85}$$

we determine the amplitudes,

$$A = x_0 \quad \text{and} \quad B = \gamma x_0 \; . \tag{4.86}$$

Finally, the solution is,

$$x(t) = x_0(1 + \gamma t)e^{-\gamma t} \; . \tag{4.87}$$

Fig. 4.32 illustrates the damping of the oscillation for various friction rates γ.

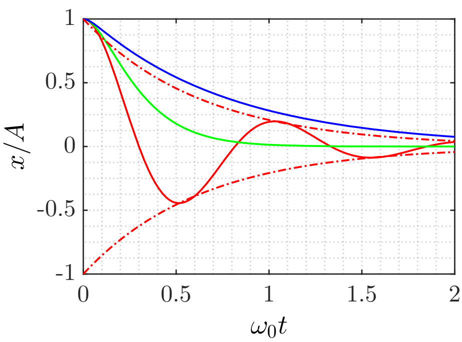

Figure 4.32: (code) Damped oscillation for $\omega_0 = 10\,\text{s}^{-1}$ and $\gamma = 2.5\,\text{s}^{-1}$ (red), $10\,\text{s}^{-1}$ (green), and $25\,\text{s}^{-1}$ (blue).

The critical friction coefficient generates a damped movement without any 'overshoot', since the velocity $\dot{x}(t)$ only disappears for $t = 0$.

4.3.1.4 Quality factor and energy loss

For a harmonic oscillation we establish the balance of energies,

$$\begin{aligned} E &= \frac{m}{2}v^2 + \frac{k}{2}x^2 \\ &= \frac{m}{2}\left(A\imath\omega_0 e^{\imath\omega_0 t} - B\imath\omega_0 e^{-\imath\omega_0 t}\right)^2 + \frac{m}{2}\omega_0^2\left(Ae^{\imath\omega_0 t} + Be^{-\imath\omega_0 t}\right)^2 = 2m\omega_0^2 AB \ . \end{aligned} \tag{4.88}$$

Now, for an underdamped oscillation we replace the amplitudes by $A \to Ae^{-\gamma t}$ and $B \to Be^{-\gamma t}$, such that,

$$E(t) = 2m\omega_0^2 ABe^{-2\gamma t} \ . \tag{4.89}$$

Obviously, the energy is decreasing at the rate 2γ.

We define the *quality factor* as the number of radians that the damped system oscillates before its energy falls to e^{-1},

$$Q = \frac{\omega}{2\gamma} = \frac{\omega m}{b} \simeq \frac{\omega_0 m}{b} \ . \tag{4.90}$$

Comparing the initial energy with the energy remaining after one cycle,

$$\frac{E}{\Delta E} = \frac{E(0)}{E(0) - E(2\pi/\omega)} = \frac{1}{1 - e^{-4\pi\gamma/\omega}} \simeq \frac{\omega}{4\pi\gamma} \ , \tag{4.91}$$

we find that the quantity,

$$\frac{Q}{2\pi} = \frac{E}{\Delta E} \tag{4.92}$$

represents a measure for the energy dissipation.

4.3.2 Forced vibration and resonance

We have seen that a damped oscillator loses its energy over time. To sustain the oscillation, it is necessary to provide energy. The simplest way to do this, is to force the oscillator to oscillate at a frequency ω by applying an external force $F_0 \cos\omega t$. The question now is, what will be the amplitude of the oscillation and its phase with respect to the phase of the applied force. We begin by establishing the equation of motion,

$$ma + bv + m\omega_0^2 x = F_0 \cos\omega t \ . \tag{4.93}$$

The calculation can be greatly simplified by the use of complex numbers. We write the differential equation as,

$$ma + bv + m\omega_0^2 x = F_0 e^{\imath\omega t} \ , \tag{4.94}$$

making the ansatz $x(t) = Ae^{\imath\omega t - \imath\delta}$, yielding

$$-\omega^2 Ae^{\imath\omega t - \imath\delta} m + \imath\omega bAe^{\imath\omega t - \imath\delta} + m\omega_0^2 Ae^{\imath\omega t - \imath\delta} = F_0 e^{\imath\omega t} \ . \tag{4.95}$$

We rewrite this formula,

$$e^{\imath\delta} = A\frac{m(\omega_0^2 - \omega^2) + \imath b\omega}{F_0} \ . \tag{4.96}$$

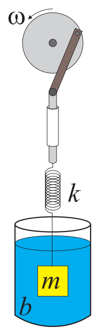

Figure 4.33: Forced oscillation damped by a viscous medium.

Immediately we get the solutions,

$$\tan\delta = \frac{\sin\delta}{\cos\delta} = \frac{\mathfrak{Im}\,e^{\imath\delta}}{\mathfrak{Re}e^{\imath\delta}} = \frac{b\omega}{m(\omega_0^2-\omega^2)} \tag{4.97}$$

$$A = \left|Ae^{-\imath\delta}\right| = \left|\frac{F_0}{m(\omega_0^2-\omega^2)+\imath\omega b}\right| = \frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2+b^2\omega^2}} \;.$$

The frequency response (spectrum) of the oscillator to the periodic excitation is illustrated in Fig. 4.34. We see that, when we increase the friction, we decrease the height and increase the width of the spectrum $|A(\omega)|$. Fig. 4.34(b) shows that, increasing the excitation frequency, the oscillation undergoes a phase shift of π.

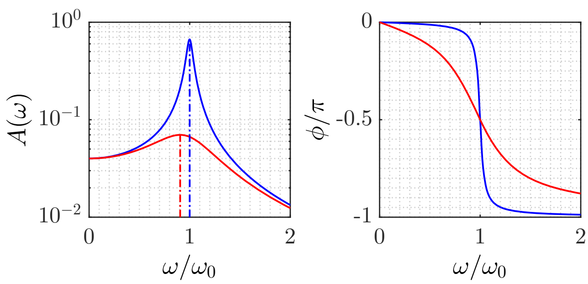

Figure 4.34: (code) Frequency response of the amplitude and phase of the oscillator for a force of $F_0 = 1\,\text{N}$, a mass of $m = 1\,\text{kg}$, a resonance frequency of $\omega_0 = (2\pi)\ 5\,\text{Hz}$, and a friction coefficient of $b = 0.5$ (blue) or $b = 1$ (red).

We now ask, at what excitation frequency ω the oscillator responds with maximum amplitude,

$$0 = \frac{d}{d\omega_m}A(\omega_m) = F_0\omega_m\frac{2m^2\omega_0^2 - 2m^2\omega_m^2 - b^2}{(m^2\omega_0^4 - 2m^2\omega_0^2\omega_m^2 + m^2\omega_m^4 + b^2\omega_m^2)^{\frac{3}{2}}} \;. \tag{4.98}$$

The numerator disappears for,

$$\omega_m = \sqrt{\omega_0^2 - \frac{b^2}{2m^2}} \;, \tag{4.99}$$

and the amplitude becomes,

$$A_m = \frac{F_0}{b\sqrt{\omega_0^2 - \frac{b^2}{4m^2}}} \,. \tag{4.100}$$

4.3.2.1 Quality factor

For weak damping $\gamma \ll \omega_0$ and small detunings, $|\omega - \omega_0| \ll \omega_0$, we can approximate the expression for the spectrum by,

$$A(\omega) \simeq \left|\frac{F_0}{m}\frac{1}{2\omega_0(\omega_0 - \omega) + \imath\omega_0\frac{b}{m}}\right| = \left|\frac{F_0}{2m\omega_0}\frac{1}{\omega - \omega_0 - \imath\gamma}\right| \,.$$

This function corresponds to a Lorentzian profile with the width FWHM $\Delta\omega = 2\gamma$. The *quality factor* defined in the section discussing the damped oscillator measures the quality of the resonance,

$$Q = \frac{\omega}{2\gamma} = \frac{\omega}{\Delta\omega} \,. \tag{4.101}$$

Example 15 (*Harmonic vibration*):

- Construct a L-C-circuit, excite it by a function generator by making a frequency ramp, and show the resonance on the oscilloscope. It works with a coil of $N = 12$ turns, of length $\ell = 6\,\mathrm{cm}$ and of radius $r = 1.4\,\mathrm{cm}$, giving $L = 1.4\,\mu\mathrm{H}$. We can also set $R = 2.2\,\Omega$ and $C = 100\,\mathrm{nF}$, giving $\omega_0 = 9.4\,\mathrm{MHz}$.

4.3.3 Exercises

4.3.3.1 Ex: Resolution of the damped oscillator equation

Solve the damped oscillator equation for $4km > b^2$ using the ansatz $x(t) = Ae^{-\gamma t}\cos\omega t$.

4.3.3.2 Ex: Damped oscillation

In a damped oscillation the oscillation period is $T = 1\,\mathrm{s}$. The ratio between two consecutive amplitudes is 2. Despite the large damping, the deviation of the period T_0 compared to the undamped oscillation is small. Calculate the deviation.

4.3.3.3 Ex: Damped physical pendulum

The physical pendulum shown in the figure consists of a disk of mass M and radius R suspended on an axes parallel to the symmetry axis of the disk and passing the edge of the disk.
a. Calculate the inertial momentum of the disk, $I = \int_V r^2 dm$, with respect to the suspension axes.
b. Derive the equation of motion by considering a weak Stokes damping due to friction proportional to the angular velocity and by approximating for small amplitude oscillations.

c. What is the natural oscillation frequency of the pendulum (without friction)? How to calculate the oscillation frequency considering friction?
d. Write down the solution of the equation of motion for the initial situation $\phi(0) = 0$ and $\dot{\phi}(0) = \dot{\phi}_0$.

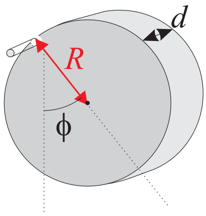

Figure 4.35: Damped physical pendulum.

4.3.3.4 Ex: Pendulum with friction

Jane has prepared dinner and Tarzan (80 kg) and Cheeta (40 kg) must return home. The house is in a tree at a height of 10 m, so that both must swing home on a (massless) rope hanging from $l = 100\,\mathrm{m}$ high tree. Tarzan grabs the rope at the height of its center-of-mass $h = 1.2\,\mathrm{m}$ above ground, Cheeta because of its height is smaller at 0.8 m above ground. With what initial speed both need to grab the rope to reach the platform of the house with their feet. Consider Stokes' friction force, $F_R = C \cdot v$ with $C = 4 \cdot 10^{-4}\,\mathrm{Ns/m}$ (Tarzan) respectively, $C = 2 \cdot 10^{-4}\,\mathrm{Ns/m}$ (Cheeta). Why is this force different for the two? Treat the oscillating motion as small displacement. Determine whether the vibration is weakly damped. Do you think Jane will have dinner alone?

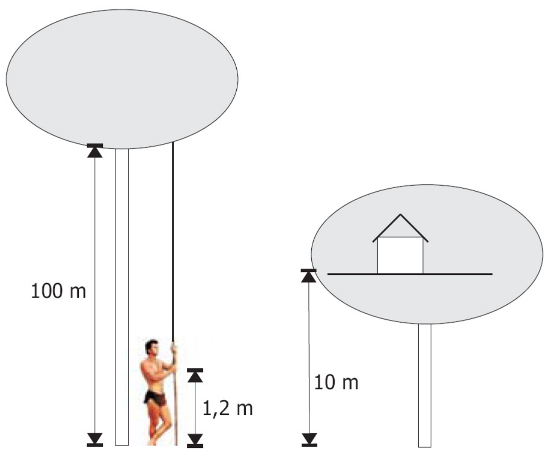

Figure 4.36: Pendulum with friction.

4.3.3.5 Ex: Oscillations and static friction

A wooden block vibrates smoothly on a horizontal spring with an oscillation period of $T = 1.2\,\mathrm{s}$. A second block of wood lies on top of the first, the coefficient of static friction between the two blocks being $\mu_H = 0.25$. The static friction is caused by the

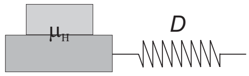

Figure 4.37: Oscillations and static friction.

surface roughness of the two blocks and depends on the mass m of the upper block like $F_H = \mu_H mg$.
a. Will the wooden block slip, if the amplitude of the vibration is 1 cm?
b. Determine the largest allowable vibration amplitude for the wooden block not to slip.

4.3.3.6 Ex: Resolution of the forced oscillator equation

Solve the forced oscillator equation using the ansatz $x(t) = A\cos(\omega t - \delta)$.

4.3.3.7 Ex: Oscillation with coercive force

On a body of mass m along the x-axis act a force proportional to the displacement $F_h = -\kappa x$ and a Stokes friction force $F_R = -\gamma\dot{x}$. A time-dependent force is switched on at time $t = 0$, while the body rests at the position $x = 0$. The force increases linearly over time until it suddenly disappears at time $t = T$. Determine the work that the external force has done up this time. Consider the various solutions of the equation of motion resulting from the various combinations of κ and γ.

4.3.3.8 Ex: Oscillation with coercive force

You want to measure the friction coefficient γ of a sphere (mass $m = 10\,\mathrm{kg}$, diameter $d = 10\,\mathrm{cm}$) in water. To do this, you let the sphere oscillate on a spring (spring constant $k = 100\,\mathrm{N/m}$) in a water bath exciting the oscillation by a periodic force, $F(t) = F_0\cos\omega t$. By varying the excitation frequency ω until observing the maximum oscillation amplitude, you measure the resonance frequency $\omega_w = 2\pi\cdot 1\,\mathrm{Hz}$. Now, you let the water out of the tub and repeat the measurement finding $\omega_0 = 2\pi\cdot 2\,\mathrm{Hz}$.
a. Determine the resting position of the mass in water and air.
b. Establish the differential equation of motion. Assume that the weight of the sphere in water is reduced by the buoyancy $V\rho_{wat}g$, where V is the volume of the sphere and ρ_{wat} the density of the water.
c. What is the value of γ?

4.3.3.9 Ex: Electronic oscillator circuit

The instantaneous current $I(t)$ in an L-R-C-circuit (inductance of a coil, ohmic resistance and capacitance in series) excited by an alternating voltage source $U(t) =$

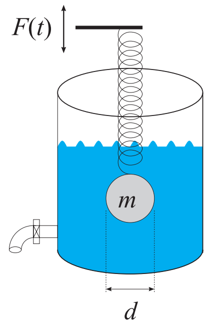

Figure 4.38: Driven pendulum.

$U_0 \cos \omega t$ satisfies the following differential equation,

$$L\dot{I} + RI + C^{-1} \int_0^t I dt' = U_0 \sin \omega t \ .$$

a. Derive the equation for the moving charge $\dot{Q} = I$, compare the obtained equation with that of the damped and forced spring-mass oscillator and determine the solution for the current.
b. Determine the resonance frequency ω_0 of the circuit.
c. Determine the quality factor Q of the circuit. How you can increase Q without changing the resonance frequency?

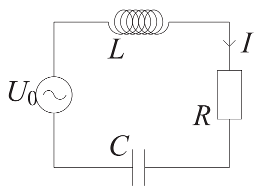

Figure 4.39: Line filter.

4.3.3.10 Ex: Electronic oscillator circuit

A voltage $U(t)$ is known to produce in an coil of inductance L the current $I_L = L^{-1} \int_0^{t'} U dt$, in an ohmic resistance R the current $I_R = R^{-1} U$, and in a capacitor of capacitance C the current $I = C\dot{U}$. In the parallel L-R-C circuit shown in the figure, at each instant of time the sum of the currents I_L, I_R and I_C must compensate the current $I_F(t) = I_0 e^{i\omega t}$ supplied by an alternating current source, while the voltage $U(t)$ is the same across all components.
a. Derive the differential equation for the derivative of the voltage $\dot{U}$.
b. What would be the oscillation frequency of the current without source ($I_0 = 0$) and without resistance ($R = \infty$)?
c. What would be the oscillation frequency of the current without source ($I_0 = 0$) but with resistance ($R \neq \infty$)?

d. Doing the ansatz $U(t) = U_0 e^{\imath\omega t + \imath\phi}$ derive the characteristic equation.
e. Use the characteristic equation to calculate the impedance defined by $Z \equiv |U_0/I_0|$ and the phase ϕ of the current oscillation as a function of the frequency ω. Prepare qualitative sketches of functions $Z(\omega)$ and $\phi(\omega)$.

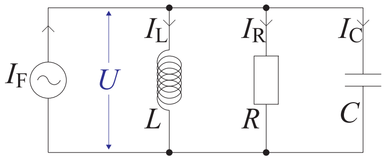

Figure 4.40: Notch filter.

4.3.3.11 Ex: Lorentz model of light-atom interaction

The Lorentz model describes the interaction of an electron attached to an atom with an incident light beam as a damped oscillator. The electron's binding to the nucleus is taken into account by a restoring force $-\omega_0^2 x$. The decay of the excited state with the rate Γ is the reason for the damping force $-m\Gamma\dot{x}$. And the excitation is produced by the Lorentz force exerted by the electrical component of the light beam, $e\mathcal{E}_0 e^{\imath\omega t}$, where e is the charge of the electron. Establish the differential equation and calculate the amplitude of electron's oscillation as a function of the excitation frequency.

4.3.3.12 Ex: Lorentz model of light-atom interaction

a. Electric fields $\mathcal{E}$ exert on electric charges q the Coulomb force $F = q\mathcal{E}$. Write the differential equation for the undamped motion of an electron (charge $-e$, mass m) harmonically bound to its nucleus under the influence of an alternating electric field, $\mathcal{E} = \mathcal{E}_0 \sin\omega t$.
b. Show that the general solution can be written as,

$$x(t) = \frac{-e\mathcal{E}_0 \sin\omega t}{m(\omega_0^2 - \omega^2)} + A\cos\omega_0 t + B\sin\omega_0 t \ .$$

c. Write the solution in terms of the initial conditions $x(0) = 0 = \dot{x}(0)$.

4.4 Coupled oscillations and normal modes

So far we have discussed the behavior of isolated oscillators. Energy losses or gains were described in a bulk way via a coupling to an external reservoir without structure of its own. However, the reservoir often has vibrational degrees of freedom, as well, and can dump (or supply) energy. This usually happens when neighboring oscillators share a rigid, massive, or sturdy medium. The transfer of energy to neighboring oscillators is the key ingredient for any oscillatory propagation of energy called *wave*.

4.4.1 Two coupled oscillators

To discuss the *coupling between oscillators* at the most fundamental level, we consider two ideal and identical pendulums (length L and mass m) coupled by a spring of constant k, as shown in Fig. 4.41. The differential equations of motion for the angles

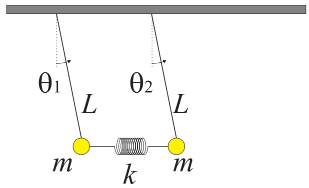

Figure 4.41: Two coupled pendulums.

θ_1 and θ_2 are,

$$mL\ddot{\theta}_1 = -mg\sin\theta_1 - k(x_1 - x_2)$$
$$mL\ddot{\theta}_2 = -mg\sin\theta_2 - k(x_2 - x_1) \ , \tag{4.102}$$

with $x_j = L\sin\theta_j$. For small oscillations we have, therefore,

$$\ddot{\theta}_1 = -\tfrac{g}{L}\sin\theta_1 - \tfrac{k}{m}(\sin\theta_1 - \sin\theta_2) \simeq -(\tfrac{g}{L} + \tfrac{k}{m})\theta_1 + \tfrac{k}{m}\theta_2$$
$$\ddot{\theta}_2 = -\tfrac{g}{L}\sin\theta_2 - \tfrac{k}{m}(\sin\theta_2 - \sin\theta_1) \simeq -(\tfrac{g}{L} + \tfrac{k}{m})\theta_2 + \tfrac{k}{m}\theta_1 \ . \tag{4.103}$$

We define the normal coordinates of the vibration $\aleph \equiv \frac{1}{\sqrt{2}}(\theta_1 - \theta_2)$ and $\Psi \equiv \frac{1}{\sqrt{2}}(\theta_1 + \theta_2)$. We find the differential equations for $\aleph$ and Ψ by adding and subtracting the equations of motion,

$$\ddot{\theta}_1 + \ddot{\theta}_2 \simeq -\tfrac{g}{L}(\theta_1 + \theta_2) \qquad \text{and} \qquad \ddot{\theta}_1 - \ddot{\theta}_2 \simeq -(\tfrac{g}{L} + \tfrac{2k}{m})(\theta_1 - \theta_2) \ ,$$

or,

$$\ddot{\Psi} + \omega_\Psi^2\Psi = 0 \qquad \text{and} \qquad \ddot{\aleph} + \omega_\aleph^2\aleph = 0$$

using the angular frequencies of the vibrational normal modes,

$$\omega_\Psi = \sqrt{\frac{g}{L}} \qquad \text{and} \qquad \omega_\aleph = \sqrt{\frac{g}{L} + \frac{2k}{m}} \ .$$

4.4.2 Normal modes

Thus, the normal coordinates Ψ and $\aleph$ allow a description of the motion by decoupled linear differential equations. A vibration involving only one *normal coordinate* is called *normal mode*. In this mode all the components participating in the oscillation oscillate at the same frequency.

The importance of the normal modes is they are totally independent, that is, they never exchange energy and they can be pumped separately. Therefore, the total energy of the system can be expressed as the sum of terms containing the squares of the normal coordinates (potential energy) and their first derivatives (kinetic energy). Every independent path by which a system can gain energy is called *degree of freedom* and has an associated normal coordinate. For example, an isolated harmonic oscillator has two degrees of freedom, as it can gain potential or kinetic energy and two normal coordinates, x and v. And the coupled oscillator system,

$$E_\aleph = a\dot{\aleph}^2 + b\aleph^2 \qquad \text{and} \qquad E_\Psi = a\dot{\Psi}^2 + b\Psi^2 \ , \tag{4.104}$$

has four degrees of freedom.[7]

Every movement of the system can be represented by a superposition of normal modes,

$$\aleph = \tfrac{1}{\sqrt{2}}(\theta_1 - \theta_2) = \aleph_0 \cos(\omega_\aleph t + \phi_\aleph) \qquad \text{and} \qquad \Psi = \tfrac{1}{\sqrt{2}}(\theta_1 + \theta_2) = \Psi_0 \cos(\omega_\Psi t + \phi_\Psi) \ . \tag{4.105}$$

Choosing $\sqrt{2}A = \aleph_0 = \Psi_0$ and $\phi_\aleph = \phi_\Psi = 0$,

$$\theta_1 = \tfrac{1}{\sqrt{2}}(\Psi + \aleph) = A\cos\omega_\aleph t + A\cos\omega_\Psi t = 2A\cos\tfrac{(\omega_\Psi - \omega_\aleph)t}{2}\cos\tfrac{(\omega_\Psi + \omega_\aleph)t}{2} \tag{4.106}$$
$$\theta_2 = \tfrac{1}{\sqrt{2}}(\Psi - \aleph) = A\cos\omega_\aleph t - A\cos\omega_\Psi t = 2A\sin\tfrac{(\omega_\Psi - \omega_\aleph)t}{2}\sin\tfrac{(\omega_\Psi + \omega_\aleph)t}{2} \ .$$

The oscillation shows the behavior of a frequency beat [8].

Example 16 (*Normal modes*):

- Two pendulums suspended on a movable horizontal bar which, in turn, is suspended by two wires to a rigid ceiling. Show (anti-)symmetric modes and their different exposure to damping of the motion of the bar.

4.4.3 Normal modes in large systems

There are techniques for solving systems many coupled oscillator. Let us consider, for example, a chain of $n = 1, ..., N$ oscillators coupled by springs. We have,

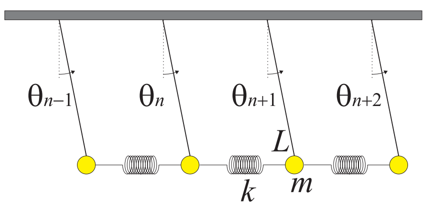

Figure 4.42: Array of coupled pendulums.

$$\ddot{\theta}_n = -\frac{g}{l}\theta_n - \frac{k}{m}(\theta_n - \theta_{n+1}) - \frac{k}{m}(\theta_n - \theta_{n-1}) \ . \tag{4.107}$$

[7]Note that the motion of a single pendulum is a movement in two Cartesian dimensions and therefore would have four degrees of freedom. However, the joint action of gravity and the tension of the wire constrains the movement into one dimension thus freezing two degrees of freedom.

[8]Normal modes are observed in the molecular vibrations of H_2O and CO_2 (see Pain).

Inserting the ansatz $\theta_n \equiv A_n e^{i\omega t}$, we obtain

$$\omega^2 A_n = \omega_0^2 A_n + \beta^2(A_n - A_{n+1}) + \beta^2(A_n - A_{n-1}) \ , \tag{4.108}$$

using the abbreviations $\omega_0^2 = g/l$ and $\beta^2 = k/m$. Defining the vector $\vec{A} \equiv (\cdots A_n \cdots)$ and the matrix,

$$\hat{M} \equiv \begin{pmatrix} \omega_0^2+\beta^2 & -\beta^2 & & & & \\ -\beta^2 & \ddots & \ddots & & & \\ & \ddots & \omega_0^2+2\beta^2 & -\beta^2 & & \\ & & -\beta^2 & \omega_0^2+2\beta^2 & \ddots & \\ & & & \ddots & \ddots & -\beta^2 \\ & & & & -\beta^2 & \omega_0^2+\beta^2 \end{pmatrix} \ , \tag{4.109}$$

we put the characteristic equation into a form called an eigenvalue equation,

$$\hat{M}\vec{A} = \omega^2\vec{A} \ . \tag{4.110}$$

The matrix $\hat{M}$ is characterized by the fact that it contains on its diagonal the energy of each individual oscillator (that is, $\omega_0^2 + 2\beta^2$ when the oscillator is in the middle of the chain, and $\omega_0^2 + \beta^2$) at the two ends of the chain). On the secondary diagonals (that is, at the positions $M_{n,n\pm1}$) are the coupling energies between two oscillators n and $n \pm 1$. A normal mode of the system corresponds to an *eigenvector* of the matrix $\hat{M}$, and the natural frequency of this mode corresponds to the respective *eigenvalue*.

The equation (4.110) has non-trivial solutions only, when the determinant of the matrix $\hat{M} - \omega^2$ vanishes. The eigenvalues are those ω^2 which satisfy this requirement,

$$\det(\hat{M} - \omega^2\mathbf{1}) = 0 \ . \tag{4.111}$$

4.4.4 Dissipation in coupled oscillator systems

We now extend the system of two coupled pendulums to include damping. Assuming that the movement of the pendulum is subject to damping,

$$\begin{aligned} \ddot{\theta}_1 &= -\Gamma\dot{\theta}_1 - \tfrac{g}{L}\theta_1 - \tfrac{k}{m}(\theta_1 - \theta_2) \\ \ddot{\theta}_2 &= -\Gamma\dot{\theta}_2 - \tfrac{g}{L}\theta_2 - \tfrac{k}{m}(\theta_2 - \theta_1) \ , \end{aligned} \tag{4.112}$$

giving the collective modes,

$$\begin{aligned} \ddot{\Psi} &= \ddot{\theta}_1 + \ddot{\theta}_2 = -\Gamma\dot{\Psi} - \tfrac{g}{L}\Psi \\ \ddot{\aleph} &= \ddot{\theta}_1 - \ddot{\theta}_2 = -\Gamma\dot{\aleph} - \left(\tfrac{g}{L} + \tfrac{2k}{m}\right)\aleph \ . \end{aligned} \tag{4.113}$$

Assuming that the movement of the spring (not the movement of the pendulums) is subject to damping,

$$\begin{aligned} \ddot{\theta}_1 &= -\tfrac{g}{L}\theta_1 - \tfrac{k}{m}(\theta_1 - \theta_2) - \Gamma(\dot{\theta}_1 - \dot{\theta}_2) \\ \ddot{\theta}_2 &= -\tfrac{g}{L}\theta_2 - \tfrac{k}{m}(\theta_2 - \theta_1) - \Gamma(\dot{\theta}_2 - \dot{\theta}_1) \ , \end{aligned} \tag{4.114}$$

giving the collective modes

$$\begin{aligned}
\ddot{\Psi} &= \ddot{\theta}_1 + \ddot{\theta}_2 = -\tfrac{g}{L}\Psi \\
\ddot{\aleph} &= \ddot{\theta}_1 - \ddot{\theta}_2 = -\left(\tfrac{g}{L} + \tfrac{2k}{m}\right)\aleph - 2\Gamma\dot{\aleph} \; .
\end{aligned} \tag{4.115}$$

Thus, the anti-symmetric mode Ψ is free from damping, while the symmetric mode $\aleph$ damps out twice as fast. Therefore, Ψ is called the *subradiant* mode and $\aleph$ the *superradiant* mode.

4.4.5 Exercises

4.4.5.1 Ex: Energy of normal modes

Verify that the total energy of a system of two coupled oscillators is equal to the sum of the energies of the normal modes.

4.4.5.2 Ex: Normal modes of two spring-coupled masses

Consider two different masses m_1 and m_2 coupled by a spring k.
a. Determine the equation of motion and the characteristic equation for each mass.
b. Write the characteristic equations in matrix form: $\hat{M}\vec{a} = \omega^2\vec{a}$, where $\vec{a} \equiv (a_1, a_2)$ and a_j are the amplitude of the oscillations and calculate the two eigenvalues of the matrix.
c. Calculate the normal modes, that is, the eigenvectors solving the equation $\hat{M}\vec{a} = \omega_k^2\vec{a}$ for each eigenvalue.
d. Derive the differential equations of the center-of-mass motion and the relative motion. Compare the result with the normal modes.

4.4.5.3 Ex: Spring-coupled chain of masses

Consider a chain of spring-coupled masses.
a. Determine the equation of motion and the characteristic equation for each mass.
b. Calculate the normal modes for a chain consisting of three masses.

4.4.5.4 Ex: Normal modes of CO_2

We consider the carbon dioxide molecule CO_2, for which we make a spring-mass model with three masses coupled by k springs in a linear chain. Calculate the frequencies of the normal modes and the eigenvectors of the vibrations.

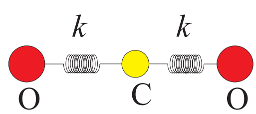

Figure 4.43: Normal modes of CO_2.

4.4.5.5 Ex: Three coupled pendulums

Determine the frequencies of the oscillation modes of a chain of three spring-coupled pendulums.

4.4.5.6 Ex: Super- and subradiance

We consider three carts attached by springs (spring constant k), as shown in the figure. The inner carts have mass m and are subject to damping by friction with the coefficient γ. The outer cart has mass M and friction Γ.
a. Establish the equations of motion of the three carts.
b. Discuss the case $M \to 0$.

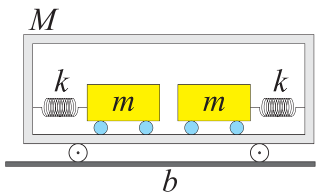

Figure 4.44: Super- and subradiant pendulums.

4.5 Further reading

H.M. Nussenzveig, Edgar Blucher (2014), *Curso de Física Básica: Fluidos, Vibrações e Ondas, Calor - vol 2* [ISBN]

Chapter 5

Waves

While in *vibrating* bodies the motion and the energy are localized in space, *waves* do propagate and carry energy to other places. In fact, waves represent the most important mechanism for transporting and exchanging energy and information. We can understand a wave as a perturbation propagating through an elastic material medium. In some cases, however, e.g. for electromagnetic waves, the propagation of the wave is due to a self-sustained oscillation between two forms of energy (electric and magnetic) without the need of a material medium. Here, is a classification of the most common types of waves: A lecture version of this chapter can be found at

Table 5.1: *Types of waves.*

wave	pulse	sound	sound	surface	light	de Broglie
medium	string	air	crystal	fluid	vacuum	particle
polarize	trans.	long.	trans./long.	long.	trans.	long.
transform	Galilei	Galilei	Galilei	Galilei	Lorentz	Galilei
wave eq.	Helmholtz	Helmholtz	Helmholtz	Helmholtz	Helmholtz	Schrödinger

(watch talk).

5.1 Propagation of waves

There are several types of wave that we will classify according to the propagation medium and to the polarization, that is, we will distinguish longitudinal and transverse waves. There are media only supporting transverse waves (strings, water surfaces). Others only withstand longitudinal waves (sound in fluid media). Finally, there are media supporting both (sound in solids, electromagnetic waves).

The simplest example of a pulse is a local deformation of a string, as shown in Fig. 5.1. The pulse travels to one end of the string by a motion called *propagation*. The propagation is not conditioned to any transport of mass, but all the particles of the system go back to their original positions after the passage of the pulse. However, there is energy transport along the string, since each of its portions suffers an increase in kinetic and potential energy during the passage of the pulse.

In general, the pulse broadens during propagation, an effect called *dispersion*. To simplify the problem let us, as a first approximation neglect the dispersion and

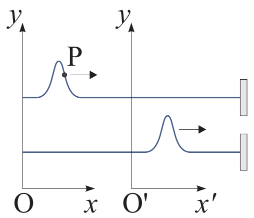

Figure 5.1: Pulse propagation along a rope.

suppose that the pulse does not change its shape,

$$Y(x,t) = f(x - vt) \ , \tag{5.1}$$

where the propagation velocity is positive when the pulse propagates in the direction of the positive x-axis.

The behavior of the pulse at the end of the rope depends on its fixation. Attached to a wall, the reflected pulse has opposite propagation amplitude and direction,

$$Y_{rfl}(x,t) = -f(x + vt) \ . \tag{5.2}$$

Fixed to another rope, the pulse will be partially reflected and partially transmitted.

5.1.1 Transverse waves, propagation of pulses on a rope

Pulses on a rope are examples for transverse waves. The speed at which the pulse propagates on a rope depends essentially on the properties of the string, that is, its mass density μ and the applied tension T, but not on the pulse amplitude. We take a small length element dx of the string with mass $dm = \mu dx$ and consider a pulse traveling with velocity v, as shown in Fig. 5.1.

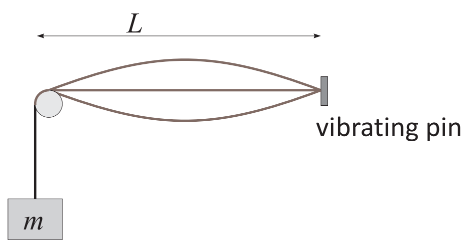

Figure 5.2: Mass element of a rope upon a passage of a pulse.

The vertical force due to the difference of tensions is,

$$F_y = T \sin\theta(x + dx) - T \sin\theta(x) \ . \tag{5.3}$$

Assuming $\theta(x)$ small, such that $\sin\theta(x) \simeq \tan\theta(x) = \frac{dY}{dx}$,

$$F_y = T \left(\frac{dY}{dx}\right)_{x+dx} - T \left(\frac{dY}{dx}\right)_x = T \frac{\partial^2 Y}{\partial x^2} dx \ . \tag{5.4}$$

On the other hand, applying Newton's second law to this string element, we find,

$$F_y = dm\frac{\partial^2 Y}{\partial t^2} \ . \tag{5.5}$$

Thus,

$$\frac{\partial^2 Y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 Y}{\partial t^2} \ . \tag{5.6}$$

This equation is called *wave equation* and fully describes the propagation of the pulse on the string. Since $Y = f(x - vt)$ depends on both x and t, the derivatives that appear in the equation are partial, that is, one derives with respect to one variable keeping the other constant. To find the velocity, we write,

$$\frac{\partial^2 Y}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial x}{\partial t}\frac{\partial Y}{\partial x}\right) = v\frac{\partial}{\partial t}\left(\frac{\partial Y}{\partial x}\right) = v\frac{\partial}{\partial x}\left(\frac{\partial Y}{\partial t}\right) = v\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial t}\frac{\partial Y}{\partial x}\right) = v^2\frac{\partial Y^2}{\partial x^2} \ , \tag{5.7}$$

and compare the second relation with the wave equation, finding,

$$v = \sqrt{\frac{T}{\mu}} \ . \tag{5.8}$$

Example 17 (*Reflection of pulses on a rope*):

- Excite a pulse on a rope fixed to the wall (i) directly or (ii) through a thinner rope.

5.1.2 Longitudinal waves, propagation of sonar pulses in a tube

Acoustic pulses are examples for longitudinal waves. They are due to a process of compression and decompression of a gaseous medium (such as air), liquid or even solid. Let us consider an oscillating piston inside a tube (cross section A) filled with air of mass density ρ_0, as shown in Fig. 5.3. When the piston moves, it causes a local pressure increase. We want to find the velocity v at which the compression travels along the tube.

As shown in Fig. 5.3, the piston causes a negative pressure gradient along the tube giving rise to an unbalanced force which accelerates mass elements of air to the right. To simplify the situation let us assume that the piston is moved with velocity u within a time interval Δt compressing the volume of the tube by a value

$$\Delta V = -Au\Delta t \ . \tag{5.9}$$

During this time, the piston accelerates a mass $m = \rho_0 V$ of air within a volume V given by the propagation velocity v of the pulse along the tube,

$$V = Av\Delta t \ . \tag{5.10}$$

The mass within this volume receives a momentum,

$$F\Delta t = mu \ . \tag{5.11}$$

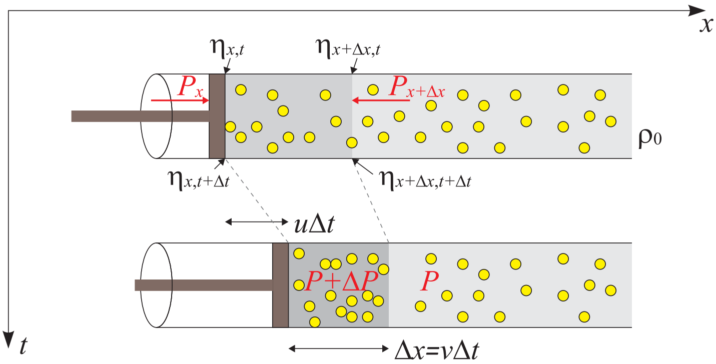

Figure 5.3: Sound waves produced by a swinging piston.

The pressure difference inside and outside the volume V causes a pressure imbalance,

$$F = A\Delta P \ . \tag{5.12}$$

With these relations we can calculate the *compressibility* of the gas,

$$\frac{1}{\kappa} \equiv -\frac{\Delta P}{\Delta V/V} = \frac{F/A}{u/v} = \frac{mu/A\Delta t}{u/v} = \frac{\rho_0 V v}{A\Delta t} = \rho_0 v^2 \ , \tag{5.13}$$

we obtain the propagation velocity of the pulse in the gas,

$$v = \sqrt{\frac{1}{\kappa\rho_0}} \ . \tag{5.14}$$

Thus, the velocity of sound propagation depends critically on the material medium. We have $v_{ar} = 331\,\text{m/s}$, $v_{H_2} = 1286\,\text{m/s}$, $v_{H_2O} = 331\,\text{m/s}$, $v_{rubber} = 54\,\text{m/s}$, and $v_{Al} = 5100\,\text{m/s}$.

To derive the equation of motion, we consider a thin gas element with thickness Δx and mass $m = \rho_0 A\Delta x$ subject to a difference of pressure on both sides of,

$$P_x - P_{x+\Delta x} = -\frac{\partial P_x}{\partial x}\Delta x = -\frac{\partial}{\partial x}(P_0 + \Delta P)\Delta x = -\frac{\partial \Delta P}{\partial x}\Delta x \ , \tag{5.15}$$

where we subtracted the background pressure P_0 assumed to be constant. This pressure difference creates a force $F = A(P_x - P_{x+\Delta x})$ accelerating the gas element following Newton's law, $F = m\ddot{\eta}$, where $\eta(x)$ is the displacement of the element, such that,

$$\frac{\partial \eta}{\partial x} = \frac{\Delta V}{V} \tag{5.16}$$

and the compression (see Fig. 5.3). We therefore obtain,

$$\rho_0 \Delta x \frac{\partial^2 \eta}{\partial t^2} = \frac{F}{A} = -\frac{\partial \Delta P}{\partial x}\Delta x \ . \tag{5.17}$$

Substituting ΔP by the relationship (5.13),

$$\rho_0 \frac{\partial^2 \eta}{\partial t^2} = -\frac{\partial}{\partial x}\left(-\frac{1}{\kappa}\frac{\partial \eta}{\partial x}\right) = \frac{1}{\kappa}\frac{\partial^2 \eta}{\partial x^2} \ , \tag{5.18}$$

which gives the wave equation. Solve Excs. 5.1.7.1, 5.1.7.2, and 5.1.7.3.

5.1.3 Electromagnetic waves

Electromagnetic waves are in several aspects different from mechanical longitudinal or transverse waves. For example, they do not need a propagation medium, but move through the vacuum at an extremely high speed. The speed of light, $c = 299792458\,\mathrm{m/s}$ exactly, is so high, that the laws of classical mechanics are no longer valid, but must be replaced by relativistic laws. And since there is no propagation medium, with respect to vacuum all inertial systems are equivalent, which will have important consequences for the Doppler effect. We will show that the electromagnetic wave equation almost comes out as a corollary of the theory of special relativity.

Electromagnetic waves always arise when a charge changes position. In this way the theory of electromagnetic waves is also a consequence of the theory electromagnetism, which is contained in Maxwell's equations. We will introduce here, without derivation, the wave equation for the electric and magnetic fields.

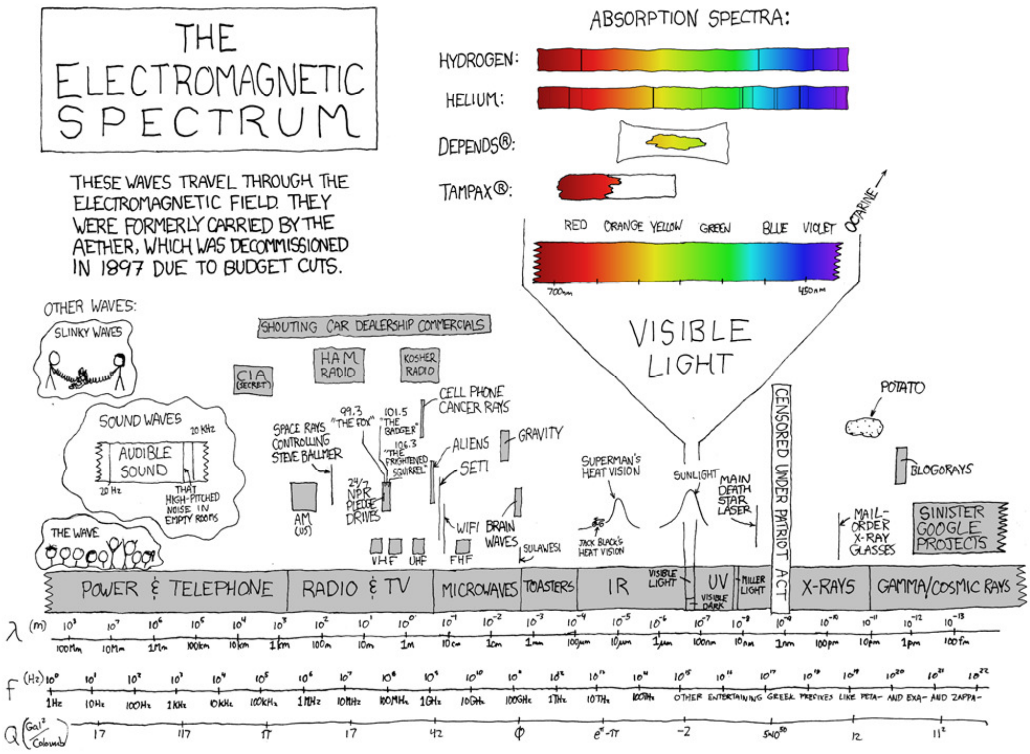

Figure 5.4: The electromagnetic spectrum.

5.1.3.1 Helmholtz equation

We have already seen how the periodic conversion between kinetic and potential energy in a pendulum can propagate in space when the pendulum is coupled to other pendulums attached to each other in a chain, and that this model explains the propagation of a pulse on the string. We also discussed how electrical and magnetic energy can be interconverted in an electronic L-C-circuit with a capacitor storing electrical energy and an inductance (a coil) storing magnetic energy. The law of electrodynamics describing the transformation of electric field variations into magnetic energy is *Ampère's law*, and the law describing the transformation of magnetic field variations into electric energy is *Faraday's law*,

$$\frac{\partial \vec{\mathcal{E}}}{\partial t} \curvearrowright \vec{\mathcal{B}}(t) \quad , \quad \frac{\partial \vec{\mathcal{B}}}{\partial t} \curvearrowright -\vec{\mathcal{E}}(t) \; . \tag{5.19}$$

Extending the circuit L-C to a chain, it is possible to show that the electromagnetic oscillation propagates along the chain. This model describes well the propagation of electromagnetic energy along a coaxial cable or the propagation of light in free space.

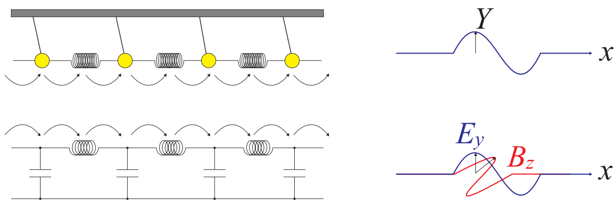

Figure 5.5: Analogy between the propagation of mechanical waves (above) and electromagnetic waves (below).

The *electrical energy* stored in the capacitor and the *magnetic energy* stored in the coil are given by,

$$E_{ele} = \tfrac{\varepsilon_0}{2}|\vec{\mathcal{E}}|^2 \quad , \quad E_{mag} = \tfrac{1}{2\mu_0}|\vec{\mathcal{B}}|^2 \; , \tag{5.20}$$

where the constants $\varepsilon_0 = 8.854 \cdot 10^{-12}$ As/Vm and $\mu_0 = 4\pi \cdot 10^{-7}$ Vs/Am are called *permittivity* and *permeability* of the vacuum. By analogy with the waves on a string, we can write the wave equations (called *Helmholtz equations*) for plane electromagnetic waves propagating along the x-axis,

$$\frac{\partial^2 \mathcal{E}_y}{\partial t^2} = \frac{1}{\varepsilon_0\mu_0}\frac{\partial^2 \mathcal{E}_y}{\partial x^2} \quad , \quad \frac{\partial^2 \mathcal{B}_z}{\partial t^2} = \frac{1}{\varepsilon_0\mu_0}\frac{\partial^2 \mathcal{B}_z}{\partial x^2} \; . \tag{5.21}$$

The formal derivation must be made from the *Maxwell equations*, which are the fundamental equations of the theory of electrodynamics. Here, we only note that,

- electromagnetic waves (in free space) are transverse;
- the electric field vector, the magnetic field vector, and the direction of propagation are orthogonal;
- the propagation velocity is the speed of light, because $c^2 = 1/\varepsilon_0\mu_0$.

5.1.3.2 Radiation intensity

In electrodynamic theory the energy flux is calculated by the *Poynting vector*,

$$\vec{S}(\mathbf{r},t) = \tfrac{1}{\mu_0}\vec{\mathcal{E}}(\mathbf{r},t) \times \vec{\mathcal{B}}(\mathbf{r},t) \; . \tag{5.22}$$

The absolute value is the intensity of the light field,

$$I(\mathbf{r},t) = |\mathbf{S}(\mathbf{r},t)| \ . \tag{5.23}$$

5.1.4 Harmonic waves

In general, a light field is a superposition of many waves with many different frequencies and polarizations and propagating in many directions. The laser is an exception. Being monochromatic, polarized, directional, and coherent, it is very close to the ideal of an *harmonic wave*, that is, a wave described by the function,

$$Y(x,t) = Y_0 \cos(kx - \omega_0 t) \ , \tag{5.24}$$

where $\omega_0 = 2\pi\nu$ is the angular frequency of the oscillation and $k = 2\pi/\lambda$ the wavevector. By inserting this function into the *wave equation*,

$$\frac{\partial^2 Y}{\partial t^2} = c^2 \frac{\partial Y^2}{\partial x^2} \ , \tag{5.25}$$

where we now call c the propagation velocity of the harmonic wave, we verify the *dispersion relation*,

$$\omega = ck \ . \tag{5.26}$$

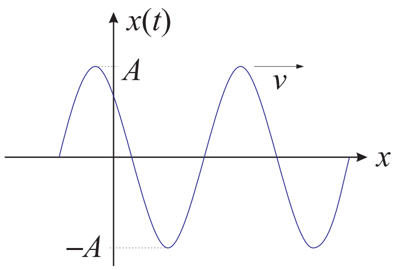

Figure 5.6: Illustration of a harmonic wave.

Often, the propagation velocity is independent of the wavelength, $c(k) = const$. In this case, a wave composed of several waves with different wavevectors k propagates without dispersing, that is, without changing its shape. In other cases, when $c(k) \neq const$, the wave deforms along its path.

5.1.5 Wave packets

Since the wave equation (5.25) is linear, the superposition principle is valid, that is, if Y_1 and Y_2 are solutions, then $\alpha Y_1 + \beta Y_2$ also is. More generally, we can say that, if $A(k)e^{\imath(kx-\omega t)}$ is a solution satisfying the wave equation for any k, then obviously,

$$Y(x,t) = \int_{-\infty}^{\infty} A(k)e^{\imath(kx-\omega t)}dk \ , \tag{5.27}$$

is. This means that the displacement $Y(x)$ and the distribution of amplitudes $A(k)$ are related by Fourier transform, $Y(x,t) = e^{-\imath\omega t}\mathcal{F}A(k)$.

Assuming a Gaussian distribution of wavevectors characterized by the width Δk, $A(k) = e^{-(k-k_0)^2/2\Delta k^2}$, we obtain as solution for the wave equation [1],

$$\begin{aligned} Y(x,t) &= \int_{-\infty}^{\infty} e^{-(k-k_0)^2/2\Delta k^2} e^{\imath(kx-\omega t)} dk \\ &= e^{\imath(k_0x-\omega t)} \int_{-\infty}^{\infty} e^{-q^2/2\Delta k^2} e^{\imath qx} dq = \sqrt{2\pi k} e^{-\Delta k^2 x^2/2} e^{\imath(k_0x-\omega t)} \ . \end{aligned} \tag{5.28}$$

This solution of the wave equation describes an *wave packet* with a Gaussian envelope [2], that is, a localized perturbation, as we discussed at the initial example of a pulse propagating on a string. Obviously, other distributions of wavevectors are possible.

Note that the width of the distribution of wavevectors, Δk, and that of the spatial distribution, $\Delta x \equiv 1/\Delta k$ satisfy a relation called *Fourier's theorem*,

$$\Delta x \Delta k = 1 \ , \tag{5.29}$$

which in quantum mechanics turns into *Heisenberg's uncertainty relation*: The broader a wavevector distribution, the narrower the spatial distribution, and vice versa. In the limit of a sinusoidal wave described by a single wavevector, we expect a infinite spatial extension of the wave.

5.1.6 Dispersion

We consider a superposition of two waves,

$$\begin{aligned} Y_1(x,t) + Y_2(x,t) &= a\cos(k_1x - \omega_1 t) + a\cos(k_2x - \omega_2 t) \\ &= 2a\cos\left[\tfrac{(k_1-k_2)x}{2} - \tfrac{(\omega_1-\omega_2)t}{2}\right]\cos\left[\tfrac{(k_1+k_2)x}{2} - \tfrac{(\omega_1+\omega_2)t}{2}\right] \ . \end{aligned} \tag{5.30}$$

The resulting wave can be regarded as a wave of frequency $\frac{1}{2}(\omega_1+\omega_2)t$ and wavelength $\frac{1}{2}(k_1+k_2)$, whose amplitude is modulated by an envelope of frequency $\frac{1}{2}(\omega_1-\omega_2)t$ and wavelength $\frac{1}{2}(k_1-k_2)x$.

---

[1] Δk is half the *total* Gaussian width at *rms* (root-mean-square) height, that is, at $1/\sqrt{e}$ of the maximum.

[2] The definition of the Fourier transform in one dimension is,

$$Y(x) = \mathcal{F}A(k) \equiv \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)e^{\imath kx} dk \ .$$

For the Gaussian function we have,

$$\begin{aligned} Y(x) &= \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ak^2} e^{\imath kx} dk = \tfrac{1}{\sqrt{2\pi}} e^{-x^2/4a} \int_{-\infty}^{\infty} e^{-a(k-ix/2a)^2} dk \\ &= \tfrac{1}{\sqrt{2\pi}} e^{-x^2/4a} \int_{-\infty}^{\infty} e^{-aq^2} dq = \tfrac{1}{\sqrt{2a}} e^{-x^2/4a} \ . \end{aligned}$$

In the absence of dispersion the *phase velocities* of the two waves and the propagation velocity of the envelope, called *group velocity*, are equal,

$$c = \frac{\omega_1}{k_1} = \frac{\omega_2}{k_2} = \frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{\Delta\omega}{\Delta k} = v_g \ . \tag{5.31}$$

However, the phase velocities of the two harmonic waves can also be different, such that the frequency depends on the wavelength, $\omega = \omega(k)$. In this case, the phase velocity also varies with the wavelength,

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}(kc) = c + k\frac{dc}{dk} \ . \tag{5.32}$$

Often this variation is not very strong, such that it is possible to expand,

$$\omega(k) = \omega_0 + \left.\frac{d\omega}{dk}\right|_{k_0} \cdot (k-k_0) + \frac{1}{2}\left.\frac{d^2\omega}{dk^2}\right|_{k_0} \cdot (k-k_0)^2 \equiv \omega_0 + v_g(k-k_0) + \beta(k-k_0)^2 \ . \tag{5.33}$$

In general we have, $v_g < c$, a situation that is called *normal dispersion*. But there are examples of *abnormal dispersion*, where $v_g > c$, e.g. close to resonances or with matter waves characterized by a quadratic dispersion relation $\hbar\omega = (\hbar k)^2/2m$.

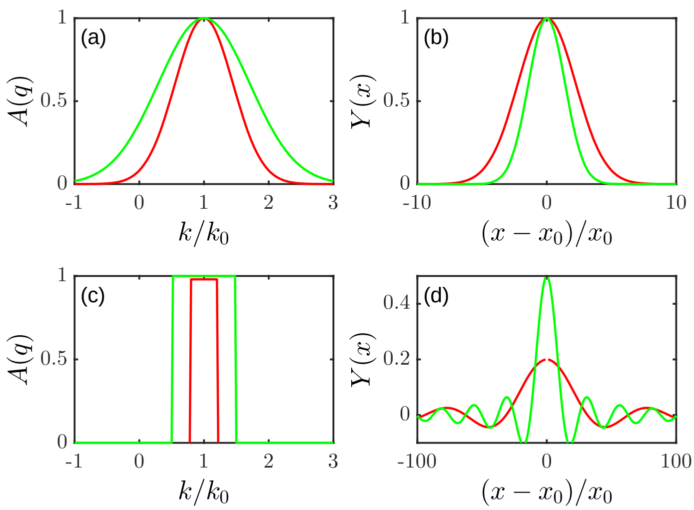

Figure 5.7: (code) Gaussian (upper graphs) and rectangular (lower graphs) distribution of amplitudes in momentum space (left) and in position space (right).

5.1.6.1 Rectangular wave packet with linear dispersion

As an example, we determine the shape of the wavepacket for a rectangular amplitude distribution, $A(k) = A_0\chi_{[k_0-\Delta k/2, k_0+\Delta k/2]}$, subject to linear dispersion (expansion up to the linear term in Eq. (5.33)). By the Fourier theorem,

$$Y(x,t) = \int_{-\infty}^{\infty} A(k)e^{\imath(kx-\omega t)}dk = A_0 \int_{k_0-\Delta k/2}^{k_0+\Delta k/2} e^{\imath(kx-\omega_0 t+\frac{d\omega}{dk}|_{k_0}(k-k_0)t)}dk \tag{5.34}$$
$$= A_0 e^{\imath(k_0x-\omega_0 t)} \int_{k_0-\Delta k/2}^{k_0+\Delta k/2} e^{\imath(k-k_0)\left(x-\frac{d\omega}{dk}|_{k_0}t\right)}dk$$
$$= A_0 e^{\imath(k_0x-\omega_0 t)} \int_{-\Delta k/2}^{\Delta k/2} e^{\imath k\left(x-\frac{d\omega}{dk}|_{k_0}t\right)}dk = A_0 e^{\imath(k_0x-\omega_0 t)} \int_{-\Delta k/2}^{\Delta k/2} e^{\imath ku}dk$$
$$= A_0 e^{\imath(k_0x-\omega_0 t)} \frac{e^{\imath\Delta k/2u} - e^{-i\Delta k/2u}}{\imath u} = 2A_0 e^{\imath(k_0x-\omega_0 t)} \frac{\sin\frac{u\Delta k}{2}}{u} \equiv A(x,t)e^{\imath(k_0x-\omega_0 t)} .$$

With the abbreviation $u \equiv x - \frac{d\omega}{dk}\big|_{k_0} t = x - v_g t$ the interpretation of the group velocity becomes obvious,

$$v_g \equiv \frac{d\omega}{dk}\bigg|_{k_0} t . \tag{5.35}$$

The envelope has the shape of a 'sinc' function, such that the intensity of the wave is,

$$|Y(x,t)|^2 = A_0\Delta k \text{ sinc}\left[\tfrac{\Delta k}{2}(x-v_g t)\right] . \tag{5.36}$$

Obviously, the *wavepacket* is localized in space. It moves at group velocity, but does not diffuse.

5.1.6.2 Dispersion of a Gaussian wave packet subject to quadratic dispersion

Quadratic dispersion leads to a spreading of the wavepackets. We show this at the example of the Gaussian wavepacket $A(k) = e^{-\alpha(k-k_0)^2}$, expanding the dispersion relation (5.33) up to the quadratic term. By the Fourier theorem,

$$Y(x,t) = \int_{-\infty}^{\infty} A(k)e^{\imath(kx-\omega t)}dk = A_0 e^{\imath(k_0x-\omega_0 t)} \int_{-\infty}^{\infty} e^{\imath(k-k_0)(x-v_g t)-(\alpha+\imath\beta t)(k-k_0)^2}dk$$
$$= A_0 e^{\imath(k_0x-\omega_0 t)} \int_{-\infty}^{\infty} e^{\imath k(x-v_g t)-(\alpha+\imath\beta t)k^2}dk$$
$$\equiv A_0 e^{\imath(k_0x-\omega_0 t)} \int_{-\infty}^{\infty} e^{\imath ku-vk^2}dk = A_0\sqrt{\tfrac{\pi}{v}}e^{\imath(k_0x-\omega_0 t)}e^{-u^2/4v} . \tag{5.37}$$

The absolute square of this solution describes the spatial energy distribution of the wavepacket,

$$|Y(x,t)|^2 = A_0^2\frac{\pi}{\sqrt{vv^*}}e^{-u^2/4v-u^2/4v^*} = A_0^2\frac{\pi}{x_0\sqrt{\alpha/2}}e^{-(x-v_g t)^2/x_0^2} , \tag{5.38}$$

with $x_0 \equiv \sqrt{2\alpha}\sqrt{1+\frac{\beta^2}{\alpha^2}t^2}$. Obviously, for long times the pulse spreads out at constant speed. Since the constant α gives the initial width of the pulse, we realize that an initially compressed pulse spreads faster. Therefore, the angular coefficient of the dispersion relation determines the group velocity, while the curvature determines the spreading speed (dispersion).

5.1.7 Exercises

5.1.7.1 Ex: Speed of sound

A person drops a stone from the top of a bridge and hears the sound of the stone hitting the water after $t = 4\,\mathrm{s}$.
a. Estimate the distance between the bridge and the water level, assuming that the propagation time of sound is negligible.
b. Improve the estimate by taking into account the finite speed of sound.

5.1.7.2 Ex: Distance of a lightning

An approximate method for estimating the distance of a lightning consists in starting to count the seconds when the lightning stroke and stop counting when the thunder arrives. The number of seconds counted divided by 3 gives the distance from the lightning in kilometers. Estimate accuracy of this procedure.

5.1.7.3 Ex: Speed of sound

A student in her room listens to the radio broadcasting a nearby football game. She is 1.6 km south of the field. On the radio, the student hears the noise generated by an electromagnetic pulse caused by a lightning strike. Two seconds later she hears the noise of thunder on the radio which was captured by the microphone of the football field. Four seconds after hearing the noise on the radio, she hears the noise of the thunder directly. Where did the lightning strike in relation to the soccer field?

5.1.7.4 Ex: Absence of dispersion in sound

Discuss the experimental evidence that leads us to assume that the speed of sound in the audible range must be the same at all wavelengths.

5.1.7.5 Ex: Optical dispersion

a. While vacuum is strictly dispersionless, the refractive index of air depends on the wavelength of light λ, on temperature T in $^\circ$ C and on the atmospheric pressure P in mbar like,

$$n_s = 1 + 10^{-8}\left(8342.13 + \frac{2406030}{130 - 10^{12}/\lambda^2} + \frac{15997}{38.9 - 10^{12}/\lambda^2}\right)$$
$$n = 1 + (n_s - 1)\frac{0.00185097P}{1 + 0.003661T} \ .$$

Calculate the dispersion of air within range $\lambda_1 = 400\,\mathrm{nm}$ and $\lambda_2 = 800\,\mathrm{nm}$.
b. Using Snell's law,

$$\frac{n_1}{n_2} = \frac{\sin\alpha_2}{\alpha_1} \ ,$$

calculate the angular dispersion $d\alpha_{ar}/d\lambda$ of a beam of light at the interface between vacuum and atmospheric air for $P = 1013\,\mathrm{mbar}$ and $T = 25^\circ$ C around $\lambda = 500\,\mathrm{nm}$.

5.1.7.6 Ex: Dispersion near an atomic resonance

Near an atomic resonance ω_0 the refractive index can be approximated by,

$$n = 1 - \frac{\alpha}{\omega^2 - \omega_0^2} ,$$

where the polarizability of the gas α is a constant. Calculate the group velocity $v_g(\omega_l)$ of a laser wave packet passing through a gas of these atoms as a function of the laser frequency ω_l. Approximate $|\omega - \omega_0| \ll \omega_0$. Make a qualitative chart of $n(\omega_l)$, $k(\omega_l)$, of the phase velocity $v_f(\omega_l)$, and of the group velocity $v_g(\omega_l)$.

5.1.7.7 Ex: Group velocity near a broad transition

The refractive index of a dilute gas (density ρ) of atoms excited by a light beam of frequency ω near a transition (resonant frequency ω_0 and width Γ) can be approximated by,

$$n = \sqrt{1 - \frac{4\pi\rho\Gamma}{k_0^3(2\Delta + \imath\Gamma)}} \simeq 1 - \frac{2\pi\rho\Gamma}{k_0^3(2\Delta + \imath\Gamma)} ,$$

where $ck_0 = \omega_0$ and $\Delta \equiv \omega - \omega_0$. Calculate the group velocity near resonance.

5.1.7.8 Ex: Dispersion in a metal

The dispersion ratio in metals can be approximated by,

$$n^2(\omega) = 1 + \omega_p^2 \left(\frac{f_e}{-\omega^2 - \imath\gamma_e\omega} + \sum_j \frac{f_j}{\omega_{0j}^2 - \omega^2 - \imath\gamma_j\omega} \right) ,$$

where ω_p is called the plasma frequency and f_e and f_j are constants. Calculate the group velocity $v_g(\omega)$.

5.2 The Doppler effect

5.2.1 Sonic Doppler effect

Waves propagate from a source to an listener within an elastic material medium with the propagation velocity v. So far, we assumed the source, the medium, and the listener at rest. The question now is, what happens when one of these three components gets in motion.

5.2.1.1 Source in motion

We imagine a source emitting signals at frequency f_0. Within the time of a period $T = \frac{1}{f_0}$ these pulses travel a distance,

$$\lambda = vT = \frac{v}{f_0} , \tag{5.39}$$

within the medium. While the source is at rest, the distance between the pulses is λ. However, when the source moves in the propagation direction of the pulses, a resting listener judges that the pulses are emitted within the medium at reduced distances Δx, as shown in Fig. 5.8,

$$\Delta x = \lambda - u_s T \ . \tag{5.40}$$

A listener now receives the pulses at the increased frequency of,

$$f = \frac{v}{\Delta x} = \frac{v}{\lambda - u_s T} = \frac{v f_0}{v - u_s} = \frac{f_0}{1 - u_s/v} \ . \tag{5.41}$$

This effect is called *sonic Doppler effect*. For small velocities we can expand,

$$f = \frac{f_0}{1 \mp u_s/v} \simeq f_0 \left(1 \pm \frac{u_s}{v}\right) \ , \tag{5.42}$$

where the upper (lower) signals apply, when the source approaches (moves away from) the listener.

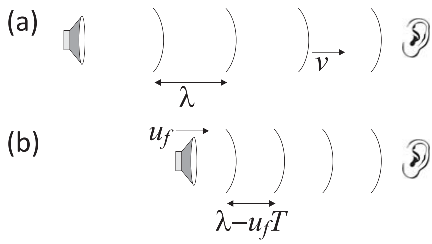

Figure 5.8: Doppler effect due to a motion of the source. In (a) the source is at rest, in (b) it moves toward the listener.

5.2.1.2 Listener in motion

Again, we consider the same source emitting signals at frequency f_0. While the source is at rest, the distance between the pulses is λ. However, when the listener is approaching the source, as shown in Fig. 5.9, pulses are recorded by the listener in a shorter time intervals,

$$T = \frac{\lambda}{v + u_r} = \frac{1}{f} \ . \tag{5.43}$$

That is, the listener measures a larger number of pulses,

$$f = f_0 \left(1 \pm \frac{u_r}{v}\right) \ , \tag{5.44}$$

where the upper (lower) signs apply, when the receiver approaches (moves away from) the source.

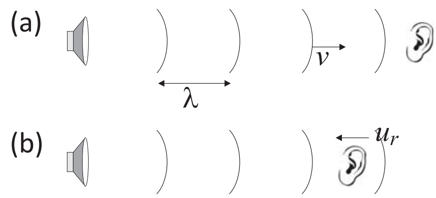

Figure 5.9: Doppler effect due to a motion of the listener. In (a) the listener is at rest, in (b) it moves toward the source.

5.2.1.3 Moving medium

We can combine the two Doppler effects into a single expression,

$$f = f_0 \frac{v^2 - \mathbf{v} \cdot \mathbf{u}_r}{v^2 - \mathbf{v} \cdot \mathbf{u}_s} \ . \tag{5.45}$$

The cases discussed above refer to the source or the listener being in motion *with respect to the medium carrying the wave* considered at rest. If the medium is moving at a velocity u_m, e.g. due to a wind moving the air, the velocities of the source and the listener with respect to the medium are modified, $u_s \to u_s - u_m$ and $u_r \to u_r - u_m$, such that,

$$f = f_0 \frac{1 - (u_r - u_m)/v}{1 - (u_s - u_m)/v} \ . \tag{5.46}$$

The same result is obtained by a transformation of the propagation velocity of the sound, $v \to v + u_m$.

5.2.2 Wave equation under Galilei transformation

The Galilei transformation says, that we obtain the function describing the motion in the system S' simply by substituting $x \to x'$ and $t \to t'$ with [3],

$$\begin{aligned} t' &\equiv t \quad \text{and} \quad x' \equiv x - ut \quad \text{or} \\ t &\equiv t' \quad \text{and} \quad x \equiv x' + ut \ , \end{aligned} \tag{5.47}$$

which implies,

$$v' = \frac{\partial x'}{\partial t'} = \frac{\partial x}{\partial t} - u = v - u \ . \tag{5.48}$$

Newton's classical mechanics is *Galilei invariant*, which means that fundamental equations of the type,

$$m\dot{v}_i = -\nabla_{x_i} \sum_j V_{ij}(|x_i - x_j|) \ , \tag{5.49}$$

[3] Note that the Galilei transform,

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = G \begin{pmatrix} ct \\ x \end{pmatrix} \quad \text{with} \quad G \equiv \begin{pmatrix} 1 & 0 \\ -\beta & 1 \end{pmatrix}$$

is unitary because $\det G = 1$.

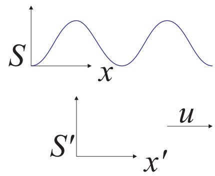

Figure 5.10: Wave in the inertial system S seen by an observer moving at the velocity u in the system S'.

do not change their form under *Galilei transform*. In contrast, the wave equation is not Galilei invariant. To see this, we consider a wave in the inertial system S being at rest with respect to the propagation medium. The wave is described by $Y(x,t)$ and satisfies the wave equation,

$$\frac{\partial^2 Y(x,t)}{\partial t^2} = c^2 \frac{\partial^2 Y(x,t)}{\partial x^2} \ . \tag{5.50}$$

An observer be in the inertial system S' moving with respect to S with velocity u, such that $x' = x - ut$. The question now is, what is the equation of motion for this wave described by $Y'(x',t')$, that is, we want to check the validity of,

$$\frac{\partial^2 Y'(x',t')}{\partial t'^2} \stackrel{?}{=} c^2 \frac{\partial^2 Y'(x',t')}{\partial x'^2} \ . \tag{5.51}$$

For example, the wave $Y(x,t) = \sin k(x - ct)$ traveling to the right is perceived in the system S', also traveling to the right, as $Y'(x',t') = \sin k[x' - (c-u)t'] = Y(x,t)$. Hence,

$$Y'(x',t') = Y(x,t) \ , \tag{5.52}$$

that is, we expect that the laws valid in S are also valid in S'. We calculate the partial derivatives,

$$\begin{aligned}
\frac{\partial Y'(x',t')}{\partial t'} &= \frac{\partial Y(x,t)}{\partial t'} = \frac{\partial t}{\partial t'} \left.\frac{\partial Y(x,t)}{\partial t}\right|_{x=const} + \frac{\partial x}{\partial t'} \left.\frac{\partial Y(x,t)}{\partial x}\right|_{t=const} = \frac{\partial Y(x,t)}{\partial t} + u\frac{\partial Y(x,t)}{\partial x} \\
\frac{\partial Y'(x',t')}{\partial x'} &= \frac{\partial Y(x,t)}{\partial x'} = \frac{\partial t}{\partial x'} \left.\frac{\partial Y(x,t)}{\partial t}\right|_{x=const} + \frac{\partial x}{\partial x'} \left.\frac{\partial Y(x,t)}{\partial x}\right|_{t=const} = \frac{\partial Y(x,t)}{\partial x} \ .
\end{aligned} \tag{5.53}$$

Therefore, we come to the conclusion that in the system propagating with the wave, the wave equation is modified,

$$\begin{aligned}
\frac{\partial^2 Y'(x',t')}{\partial t'^2} &= \frac{\partial^2 Y(x,t)}{\partial t^2} + 2u\frac{\partial^2 Y(x,t)}{\partial t \partial x} + u^2 \frac{\partial^2 Y(x,t)}{\partial x^2} \\
&= c^2 \frac{\partial^2 Y(x,t)}{\partial x^2} + 2u\frac{\partial^2 Y(x,t)}{\partial t \partial x} + u^2 \frac{\partial^2 Y(x,t)}{\partial x^2} \\
&= (c^2 + u^2)\frac{\partial^2 Y(x,t)}{\partial x^2} + 2u\frac{\partial^2 Y(x,t)}{\partial t \partial x} = (c^2 - u^2)\frac{\partial^2 Y'(x',t')}{\partial x'^2} + 2u\frac{\partial^2 Y'(x',t')}{\partial t' \partial x'} \ .
\end{aligned} \tag{5.54}$$

Only in cases, where the wavefunction can be written as $Y(x,t) = f(x - ct) = f(x' - (c-u)t') = f'(x' - ct') = Y'(x',t')$, do we obtain a wave equation similar to the one of the system S, but with the modified propagation velocity,

$$\begin{aligned}\frac{\partial^2 f'(x'-ct')}{\partial t'^2} &= (c^2-u^2)\frac{\partial^2 f'(x'-ct')}{\partial x'^2} + 2u\frac{\partial^2 f'(x'-ct')}{\partial x'\partial t'} \\ &= (c^2-u^2)\frac{\partial^2 f'(x'-ct')}{\partial x'^2} + 2u\frac{\partial^2 f(x'-(c-u)t')}{\partial x'\partial t'} \\ &= (c^2-u^2)\frac{\partial^2 f'(x'-ct')}{\partial x'^2} - 2u(c-u)\frac{\partial^2 f'(x'-ct')}{\partial x'^2} = (c-u)^2\frac{\partial^2 f'(x'-ct')}{\partial x'^2} \ .\end{aligned} \tag{5.55}$$

The observation that the wave equation is not Galilei-invariant expresses the fact that there is a preferential system for the wave to propagate, which is simply the system in which the propagation medium is at rest. Only in this inertial system will a spherical wave propagate isotropically.

Example 18 (*Wave equation under Galilei transformation*): We now verify the validity of the wave equation in the propagating system S' using the example of a sine wave,

$$\begin{aligned}&(c^2-u^2)\frac{\partial^2 \sin k[x'-(c-u)t']}{\partial x'^2} + 2u\frac{\partial^2 \sin k[x'-(c-u)t']}{\partial x'\partial t'} \\ &= -k^2(c^2-u^2)\sin k[x'-(c-u)t'] + 2uk^2(c-u)\sin k[x'-(c-u)t'] \\ &= -k^2(c-u)^2\sin k[x'-(c-u)t'] = \frac{\partial^2 \sin k[x'-(c-u)t']}{\partial t'^2} \ .\end{aligned}$$

5.2.3 Wave equation under Lorentz transformation

The question now is, how about electromagnetic waves which, as we have already noted and as has been verified by the famous Michelson experiment, survive without any medium. If there is no propagation medium, all inertial systems should be equivalent and the wave equation should be the same in all systems, as well as the propagation velocity, i.e. the speed of light. These were the consideration of *Jules Henry Poincaré*. To resolve the problem we need another transformation than the one of *Galileo Galilei*. Who found it first was *Hendrik Antoon Lorentz*, however the biggest intellectual challenge was to accept all the consequences that this transformation bears. It was *Albert Einstein* who accepted the challenge and created a new mechanics called *relativistic mechanics*. As the wave equation for electromagnetic waves, called the *Helmholtz equation*, is a direct consequence of Maxwell's theory, it is not surprising that the relativistic theory is not only compatible with the electrodynamic theory, but provides a deeper understanding of it.

We begin by making the ansatz of a general transformation interconnecting the temporal and spatial coordinates by four unknown parameters, γ, $\tilde{\gamma}$, β, and $\tilde{\beta}$,

$$ct = \gamma(ct' + \beta x') \qquad \text{and} \qquad x = \tilde{\gamma}(x' + \tilde{\beta}ct') \ . \tag{5.56}$$

The same calculation made for the Galilei transform now gives the first derivatives,

$$\frac{\partial Y'(x',t')}{c\partial t'} = \frac{\partial Y(x,t)}{c\partial t'} = \frac{\partial t}{\partial t'}\frac{\partial Y(x,t)}{c\partial t}\bigg|_{x=const} + \frac{\partial x}{c\partial t'}\frac{\partial Y(x,t)}{\partial x}\bigg|_{t=const} = \gamma\frac{\partial Y(x,t)}{c\partial t} + \tilde{\gamma}\tilde{\beta}\frac{\partial Y(x,t)}{\partial x} \tag{5.57}$$

$$\frac{\partial Y'(x',t')}{\partial x'} = \frac{\partial Y(x,t)}{\partial x'} = \frac{\partial ct}{\partial x'}\frac{\partial Y(x,t)}{c\partial t}\bigg|_{x=const} + \frac{\partial x}{\partial x'}\frac{\partial Y(x,t)}{\partial x}\bigg|_{t=const} = \gamma\beta\frac{\partial Y(x,t)}{c\partial t} + \tilde{\gamma}\frac{\partial Y(x,t)}{\partial x}\ .$$

The second derivatives and the application of the wave equation in the system S give,

$$\begin{aligned}\frac{\partial^2 Y'(x',t')}{c^2\partial t'^2} &= \gamma^2\frac{\partial^2 Y(x,t)}{c^2\partial t^2} + 2\gamma\tilde{\gamma}\tilde{\beta}\frac{\partial^2 Y(x,t)}{c\partial t\partial x} + (\tilde{\gamma}\tilde{\beta})^2\frac{\partial^2 Y(x,t)}{\partial x^2} \\ &= \gamma^2\frac{\partial^2 Y(x,t)}{\partial x^2} + 2\gamma\tilde{\gamma}\tilde{\beta}\frac{\partial^2 Y(x,t)}{c\partial t\partial x} + (\tilde{\gamma}\tilde{\beta})^2\frac{\partial^2 Y(x,t)}{c^2\partial t^2} \\ &= (\gamma\beta)^2\frac{\partial^2 Y(x,t)}{c^2\partial t^2} + 2\gamma\tilde{\gamma}\beta\frac{\partial^2 Y(x,t)}{c\partial t\partial x} + \tilde{\gamma}^2\frac{\partial^2 Y(x,t)}{\partial x^2} = \frac{\partial^2 Y(x',t')}{\partial x'^2}\ .\end{aligned} \tag{5.58}$$

That is, the wave equation in the system S' has the same form [4]. Thus, the requirement of invariance of the wave equation allows to affirm,

$$\gamma = \tilde{\gamma} \qquad \text{and} \qquad (\gamma\beta)^2 = (\tilde{\gamma}\tilde{\beta})^2 \qquad \text{and} \qquad \beta = \tilde{\beta}\ . \tag{5.59}$$

In addition, the transformation

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = L\begin{pmatrix} ct \\ x \end{pmatrix} \qquad \text{with} \qquad L \equiv \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \tag{5.60}$$

has to be unitary, that is,

$$1 = \det L = \gamma\tilde{\gamma} - \gamma\tilde{\gamma}\beta\tilde{\beta} = \gamma^2(1-\beta^2)\ , \tag{5.61}$$

which allows to relate the parameters γ and β by,

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}\ . \tag{5.62}$$

Finally and obviously, we expect to recover the Galilei transform at low velocities,

$$ct = \gamma(ct' + \beta x') \rightarrow ct \qquad \text{and} \qquad x = \gamma(x' + \beta ct') \rightarrow x + ut\ . \tag{5.63}$$

That is, the limit is obtained by $\gamma \rightarrow 1$ and $\gamma\beta c \rightarrow u$, such that,

$$\beta = \frac{u}{c}\ . \tag{5.64}$$

such that the *Lorentz transform* from one inertial system S to another system S' is,

$$\begin{aligned} t' &= \gamma\left(t - \tfrac{u}{c^2}x\right) \qquad \text{and} \qquad x' = \gamma(x - ut) \qquad \text{or} \\ t &= \gamma\left(t' + \tfrac{u}{c^2}x'\right) \qquad \text{and} \qquad x = \gamma(x' + ut')\ . \end{aligned} \tag{5.65}$$

[4] Note that the calculus is dramatically simplified using the covariant formalism of 4-dimensional *space-time vectors* introduced by *Hermann Minkowski* and *Gregory Ricci-Curbastro*.

5.2.4 Relativistic Doppler effect

We have seen at the example of sonic waves, that the magnitude of the Doppler effect depends on who moves with respect to the medium, whether it is the source or the listener. Electromagnetic waves, however, propagate in empty space, hence there is no material medium or wind. According to Einstein's theory of relativity, there is no absolute motion and the propagation velocity of light is the same for all inertial systems. Therefore, the theory of the sonic Doppler effect can not apply to electromagnetic waves. To deal with the Doppler effect of light, we need to talk a little about time dilation.

5.2.4.1 Dilation of time

We consider a clock flying through the lab S with the velocity v. The clock produces regular time intervals for which we measure in the lab the duration $t_2 - t_1$. The spatio-temporal points are Lorentz-transformed to the system S' in which the clock is at rest by,

$$\begin{pmatrix} ct'_j \\ z'_j \end{pmatrix} = \begin{pmatrix} \gamma ct_j - \gamma\beta z_j \\ -\gamma\beta ct_j + \gamma z_j \end{pmatrix} . \tag{5.66}$$

Hence,

$$\begin{aligned} t'_2 - t'_1 &= \gamma t_2 - \gamma\beta\frac{z_2}{c} - \gamma t_1 + \gamma\beta\frac{z_1}{c} \\ &= \gamma t_2 - \beta\left(\frac{z'}{c} + \gamma\beta t_2\right) - \gamma t_1 + \beta\left(\frac{z'}{c} + \gamma\beta t_1\right) = \gamma^{-1}(t_2 - t_1) . \end{aligned} \tag{5.67}$$

Consequently, in the lab the time interval seems longer than in the resting system.

Example 19 (*Doppler effect on a moving laser*): Coming back to the Doppler effect we now consider a light source flying through the lab S, for example, a laser operating at a frequency ω', which is well defined by an atomic transition of the active medium. A spectrometer installed in the same resting system S' as the laser will measure just this frequency. Now we ask ourselves, what frequency would a spectrometer installed in the lab measure. The classical response has been derived for a moving sound source,

$$\omega = \omega' - ku = \omega' - \frac{\omega}{c}u = \frac{\omega'}{1 + \frac{u}{c}} , \tag{5.68}$$

with $k = \omega/c$. But now, because of time dilation, we need to multiply by γ,

$$\omega = \frac{\gamma^{-1}\omega'}{1 + \frac{u}{c}} = \sqrt{\frac{1-\beta}{1+\beta}}\omega' \simeq \omega'\left(1 \pm \frac{u}{c} + \frac{u^2}{2c^2}\right) . \tag{5.69}$$

5.2.5 Exercises

5.2.5.1 Ex: Sonic Doppler effect

A speaker hanging from a wire of length $L = 1\,\mathrm{m}$ oscillates with a maximum angle of $\theta_m = 10°$ and emits a sound of $\nu = 440\,\mathrm{Hz}$.

a. What is the frequency of oscillation of the pendulum?
b. What is the energy $E_{cin} + E_{pot}$ of the oscillation?
c. What is the maximum oscillation speed?
d. What are the minimum and maximum frequencies of the sound perceived by a stationary receiver.

5.2.5.2 Ex: Sonic Doppler effect

Two identical speakers uniformly emit sound waves of $f = 680\,\mathrm{Hz}$. The audio power of each speaker is $P = 1\,\mathrm{mW}$. A point P is $r_1 = 2.0\,\mathrm{m}$ away from one device and $r_2 = 3.0\,\mathrm{m}$ from the other.
a. Calculate the intensities I_1 and I_2 of the sound from each speaker separately at the P.
b. If the emission of the speakers were coherent and in phase, what would be the sound intensity in P?
c. If the emission of the speakers were coherent with a phase difference of $180°$, what would be the sound intensity in P?
d. If the speaker output were incoherent, what would be the sound intensity in P?

5.2.5.3 Ex: Sonic Doppler effect

Suppose that a source of sound and a listener are both at rest, but the medium is moving relative to this frame. Will there be any variation in the frequency heard by the observer?

5.2.5.4 Ex: Sonic Doppler effect

Consider a source that emits waves of frequency f_{fnt} moving at velocity v_{fnt} on the x-axis. Consider an observer moving with velocity v_{obs} also on the x-axis. What will be the frequency perceived by the observer? Call the wave propagation velocity of c.

5.2.5.5 Ex: Sonic Doppler effect

Two trains travel on rails in opposite directions at velocities of the same magnitude. One of them is whistling. The whistle frequency perceived by a passenger on the other train ranges from $348\,\mathrm{Hz}$ when approaching to $259\,\mathrm{Hz}$ when moving away.
a. What is the velocity of the trains.
b. What is the frequency of the whistle.

5.2.5.6 Ex: Sonic Doppler effect

On a mountain road, while approaching a vertical wall which the road will surround, a driver is honking his horn. The echo from the wall interferes with the sound of the horn, producing 5 beats per second. Knowing that the frequency of the horn is $200\,\mathrm{Hz}$, what is the speed of the car?

5.2.5.7 Ex: Sonic Doppler effect

A fixed sound source emits a sound of frequency ν_0. The sound is reflected by a fast approaching object (velocity u). The reflected echo returns to the source, where it interferes with the emitted waves giving rise to frequent beats $\Delta\nu$. Show that it is possible to determine the amplitude of the velocity of the moving object $|u|$ as a function of $\Delta\nu$, of ν_0, and of the speed of sound c.

5.2.5.8 Ex: Sonic Doppler effect

Two cars (1 and 2) drive in opposite directions on a road, with velocities of amplitudes v_1 and v_2. Car 1 travels against the wind, whose velocity is V. At sight of car 2 the driver of car 1 presses his horn, whose frequency is ν_0. The speed of sound in motionless air is c. What is the frequency ν of the horn sound perceived by the driver of car 2? What is the frequency ν' heard by the driver of a car 3 traveling in the same direction as car 1 and at the same speed?

5.2.5.9 Ex: Sonic Doppler effect

A physicist is molested by a fly orbiting his head. Since he is also a musician, he realizes that the sound of the buzz varies by one pitch. Calculate the speed of the fly.

5.2.5.10 Ex: Doppler effect

a. In a storm with wind velocity v a speaker well attached to the ground makes a sound of frequency f_0. How do you calculate the frequency recorded by a microphone taken by the wind and driven away from the speaker at the speed u?
b. Verify your answer in (a) by comparing the three cases (i) $u = 0$, (ii) $u = v$, and (iii) $v = 0$ with the cases of a moving source or receiver.

5.2.5.11 Ex: Sonic Doppler effect

A citizen of São Carlos is molested by a Tucano airplane operated by the Academia das Forças Aéreas de Pirassununga. He notices that while the airplane realizes looping on top of his head, the emitted sounds varies by up to an octave. Estimate the airplane's velocity.

5.3 Interference

The superposition of two counterpropagating waves can generate a *standing wave*. In these waves the oscillation amplitude depends on the position, but there is no energy transport.

5.3.1 Standing waves

We consider two waves $Y_\pm(x,t) = A\cos(kx \mp \omega t + \phi)$ propagating in opposite directions. In the case of a string this situation can be realized, e.g. by exciting a wave

$Y_-(x,t)$ propagating in $-x$ direction, reflecting it subsequently at the end of the string $(x=0)$, and letting the wave $Y_+(x,t)$ propagate back in x direction,

$$Y(x,t) = Y_-(x,t) + Y_-(x,t) = A\cos(kx+\omega t) \pm A\cos(kx-\omega t) \ . \tag{5.70}$$

The sign of the reflected wave depends on how the end of the string is attached. If the end is fixed, the reflected wave inverts its amplitude. If it is free to move, the amplitude remains unaltered.

Let L be the length of the rope. The boundary conditions can be formulated as follows: When one end is clamped, the oscillation amplitude must be zero at this end,

$$Y(0,t) = 0 \qquad \text{or} \qquad Y(L,t) = 0 \ . \tag{5.71}$$

When one end is loose, the amplitude of oscillation must be maximum,

$$Y(0,t) = A \qquad \text{or} \qquad Y(L,t) = A \ . \tag{5.72}$$

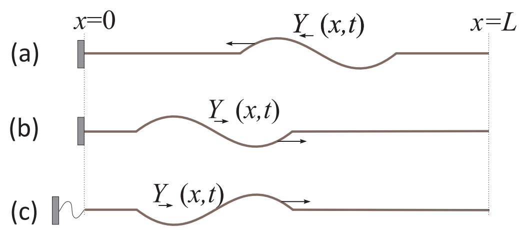

Figure 5.11: Superposition of a left-bound wave (a) with a wave reflected at a clamped end (b) or a loose end (c).

5.3.1.1 Rope with two ends fastened

In case that the two ends of the rope are clamped, we can simplify the superposition (5.70),

$$Y(x,t) = A\cos(kx+\omega t) - A\cos(kx-\omega t) = 2\sin kx \sin\omega t \ . \tag{5.73}$$

The boundary condition, $Y(L,t)=0$, requires,

$$kL = \frac{2\pi L}{\lambda} = n\pi \ , \tag{5.74}$$

for a natural number n. This means that for a given length L and a given propagation velocity v, we can only excite oscillations satisfying,

$$\lambda = \frac{2L}{n} \qquad \text{and} \qquad \nu = \frac{v}{\lambda} = n\frac{v}{2L} \ . \tag{5.75}$$

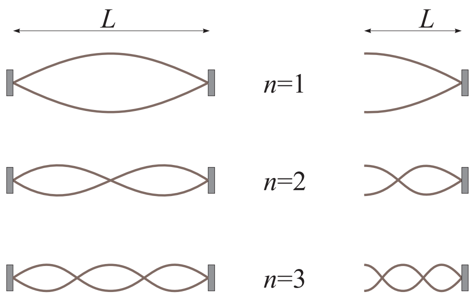

Figure 5.12: Vibration modes of a string fot (left) both ends tight up and (right) for one loose end.

5.3.1.2 Rope with one free end

In case that the end of the string at $x = 0$ is loose, we can simplify the superposition (5.70),

$$Y(x,t) = A\cos(kx+\omega t) + A\cos(kx-\omega t) = 2\cos kx\cos\omega t \ . \tag{5.76}$$

The boundary condition, $Y(L,t) = 0$, requires,

$$\phi = \tfrac{\pi}{2} \qquad \text{and} \qquad kL = \frac{2\pi L}{\lambda} = \left(n - \tfrac{1}{2}\right)\pi \ , \tag{5.77}$$

for a natural number n. This means that for a given length L and a given propagation velocity v, we can only excite oscillations satisfying,

$$\lambda = \frac{2L}{n-\frac{1}{2}} \qquad \text{and} \qquad \nu = \frac{v}{\lambda} = \left(n - \tfrac{1}{2}\right)\frac{v}{2L} \ . \tag{5.78}$$

Example 20 (*Stationary sound wave*):

- Exciting a stationary sound wave in a bottle.
- Exciting a standing sound wave on a guitar string.

5.3.2 Interferometry

5.3.2.1 Phase matching of two laser beams

When phase-matching two plane waves $\mathcal{E}_1 = Ae^{\imath\omega_1 t}$ and $\mathcal{E}_2 = Ae^{\imath\omega_2 t}$ on a photodiode, such that their wavevectors are parallel, the photodiode generates a *beat signal*,

$$I = |E_1 + E_2|^2 = AB[2 + 2\cos(\omega_1 - i\omega_2)t] \ . \tag{5.79}$$

In order to get a high signal contrast, a good phase-matching is important. It is particularly important to adjust the wavevectors to be absolutely parallel. In practice, however, this can be tricky, as the laser beams are frequently not plane waves, but have a finite diameter and radius of curvature.

Example 21 (*Laser interferometry*):

- Construct Michelson and Mach-Zehnder laser interferometers with one mirror mounted on a piezo. Show interference rings.

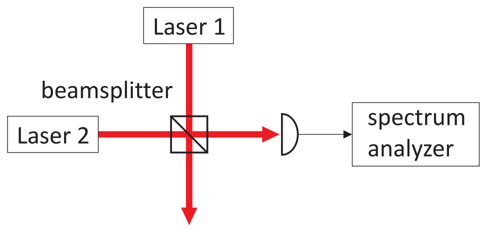

Figure 5.13: Principle of a beat frequency measurement.

5.3.3 Diffraction

According to the Huygens principle, each point P_z within a slit emits a spherical wave reaching a given point P_k of the screen with a phase lag corresponding to the distance, as shown in Fig. 5.14,

$$r_{12} = \overline{P_\eta P_y} = \sqrt{(y-\eta)^2 + z^2} \ . \tag{5.80}$$

Thus, the phase difference between this ray and a ray coming out of the origin (which we place somewhere on the optical axis) is,

$$\phi = k\Delta r_{12} = k(\sqrt{(y-\eta)^2 + z^2} - \sqrt{y^2 + z^2}) \simeq -\frac{ky\eta}{\sqrt{y^2+z^2}} \simeq -\frac{ky\eta}{z} \equiv q\eta \ , \tag{5.81}$$

with $q = k\sin\alpha = ky/z$. If $A(\eta)$ is the amplitude of the excitation at the point η of the slit, then $B(y) = \frac{1}{z}e^{\imath\phi}$ is the amplitude at point y of the screen. Adding the contributions of all points,

$$B(q) = \sum_z e^{\imath\phi(y,z)} \rightarrow \int A(\eta)e^{\imath q\eta}d\eta \ . \tag{5.82}$$

We see that the amplitude distribution on the screen $B(y)$ is nothing more than the Fourier transform of the amplitude distribution $A(\eta)$ within the slit, regardless of the shape of the slit.

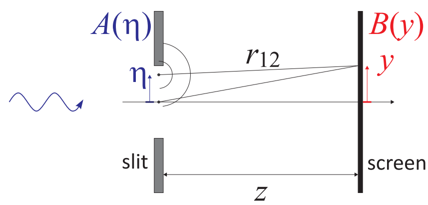

Figure 5.14: Fraunhofer diffraction at the slit.

The theory can be extended to 2D and 3D geometries, for example, a distribution of point-like scatterers within a given volume.

5.3.3.1 Single slit

As an example, we calculate the interference pattern behind a single slit. The Fourier transform of $A(\eta) = \chi_{[-d/2,d/2]}$ is,

$$B(q) = \int_{-d/2}^{d/2} e^{\imath q\eta} d\eta = \left.\frac{e^{\imath q\eta}}{\imath q}\right|_{-d/2}^{d/2} = d\frac{\sin\frac{1}{2}qd}{\frac{1}{2}qd} \ . \tag{5.83}$$

The intensity is $I(q) = c\varepsilon_0|B(q)|^2$.

5.3.3.2 Diffraction grating

We now calculate the interference pattern behind a diffraction grating with $N = 1000$ infinitely thin slits aligned within one millimeter. The Fourier transform of $A(\eta) = \sum_{n=1}^{N} \chi_{[(n-1)d,(n-1)d+\Delta d]}$ is,

$$\begin{aligned} B(q) = \sum_{n=1}^{N} \int_{(n-1)d}^{(n-1)d+\Delta d} e^{\imath q\eta} d\eta &= \frac{e^{\imath q\Delta d} - 1}{\imath q} \sum_{n=1}^{N} e^{\imath(n-1)qd} \\ &\simeq \Delta d \sum_{n=0}^{N} e^{\imath nqd} = \Delta d\frac{1 - e^{\imath Nqd}}{1 - e^{\imath qd}} \ , \end{aligned} \tag{5.84}$$

where we approximated for $q\Delta d \ll 1$. For $N \to \infty$ we can approximate further,

$$B(q) = \frac{\Delta d}{1 - e^{\imath qd}} \ . \tag{5.85}$$

This is the *Airy function*, which is zero everywhere except at points where $qd = 2n\pi$. The intensity is,

$$I(q) = c\varepsilon_0|B(q)|^2 = c\varepsilon_0\frac{\Delta d^2}{2 - 2\cos qd} = c\varepsilon_0\left(\frac{\Delta d}{2\sin\frac{qd}{2}}\right)^2 \ . \tag{5.86}$$

The grating constant is $d = 0.001\,\text{mm}$. The resulting pattern can be interpreted as arising from a regular chain of antennas emitting synchronously. With a large number of point antennas, the chain emits in very well-defined directions. In addition, the direction can be controlled by arranging for a well-defined phase shift between the fields driving neighboring antennas.

5.3.4 Plane and spherical waves

In three dimensions the wave equation takes the form,

$$0 = \Box E \equiv \left(\frac{1}{c^2}\frac{\partial}{\partial t} - \nabla^2\right) E \ . \tag{5.87}$$

In Cartesian coordinates, this gives,

$$0 = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right) E \ . \tag{5.88}$$

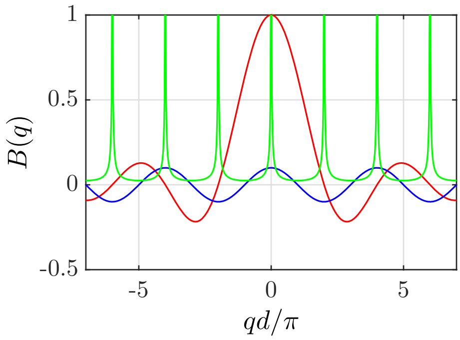

Figure 5.15: (code) Intensity distribution behind a diffraction grating for a single slit (red), a double slit (blue), and an infinite diffraction grating (green).

Plane waves, that is, waves described by the function,

$$Y(\vec{r},t) = Y_0 \sin(\vec{k}\cdot\vec{r} - \omega t) \ , \tag{5.89}$$

satisfy the wave equation if,

$$0 = -\frac{\omega^2}{c^2} + k_x^2 + k_y^2 + k_z^2 = -\frac{\omega}{c^2} + \vec{k}^2 \ . \tag{5.90}$$

5.3.4.1 Spherical waves

Spherical waves, that is, waves described by the function,

$$Y(\mathbf{r},t) = f(r)\sin(kr - \omega t) \ , \tag{5.91}$$

also satisfy the wave equation, provided the function $f(r)$ satisfies certain conditions. To find these conditions we use the representation of the *Laplace operator* in spherical coordinates,

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \ , \tag{5.92}$$

and insert the ansatz for $Y(\mathbf{r},t)$ into the wave equation. We have on one hand,

$$\frac{1}{c^2}\frac{d^2}{dt^2}(f\sin) = -\frac{\omega^2}{c^2}f\sin \ . \tag{5.93}$$

On the other hand,

$$\begin{aligned}\frac{1}{r}\frac{d^2}{dr^2}(rf\sin) &= \frac{1}{r}\frac{d}{dr}[f\sin + rf'\sin + krf\cos] \\ &= f''\sin + \frac{2f'}{r}\sin - k^2 f\sin + \frac{2k}{r}f\cos + 2kf'\cos \ ,\end{aligned} \tag{5.94}$$

such that,

$$0 = \Box f\sin = -\left(f'' + \frac{2f'}{r}\right)\sin - 2k\left(f' + \frac{f}{r}\right)\cos \ . \tag{5.95}$$

Thus the function f must satisfy the radial differential equation,

$$rf' + f = 0 \; . \tag{5.96}$$

This equation can be easily solved with the result $f(r) = r^{-1}$.

5.3.5 Formation of light beams

We consider monochromatic waves with frequency ω. Other waveforms can be synthesized by superpositions of waves with different frequencies. We also restrict to scalar waves. In fact, electromagnetic light fields are vectorial, however, close to the axis of an optical beam the fields are practically uniformly polarized, and representing the amplitude of the field by a scalar wave is an excellent approximation. The field amplitude $\psi(\mathbf{r}, t)$ is governed by the following scalar wave equation,

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} \; . \tag{5.97}$$

We let ψ be of the form,

$$\psi(\mathbf{r}, t) = A(\mathbf{r})e^{\imath[\phi(\mathbf{r})-\omega t]} \; , \tag{5.98}$$

where A and ϕ are real functions of space. A is the *amplitude*, and the exponent is called the *phase* of the wave. In this form, it is implied that abrupt spatial or temporal variations are contained in the phase. The surface obtained by fixing the phase equal to a constant,

$$\phi(\mathbf{r}) - \omega t = const \tag{5.99}$$

is called *wave front* or *phase front*. The fast motion associated with a wave can be followed through the propagation of a particular wavefront. The interference between two waves is formed by the fronts of the two waves. The speed at which a particular wavefront is moving is called *phase velocity*. Suppose we follow a particular wavefront at the moment t: At time $t + \Delta t$, the phase front will have moved to another surface. A point $\mathbf{r}$ on the original surface will have moved to another point $\mathbf{r} + \Delta\mathbf{r}$ [see Fig. 5.16(a)]:

$$\phi(\mathbf{r} + \Delta\mathbf{r}) - \omega(t + \Delta t) = \phi(\mathbf{r}) - \omega t = const \tag{5.100}$$

Expanding $\phi(\mathbf{r} + \Delta\mathbf{r}) \simeq \phi(\mathbf{r}) + \nabla\phi(\mathbf{r})\Delta\mathbf{r}$, we obtain,

$$\nabla\phi(\mathbf{r})\Delta\mathbf{r} = -\omega t \; . \tag{5.101}$$

$\nabla\phi(\mathbf{r})$ is orthogonal to the phase front and is called the *wavevector*. $\Delta\mathbf{r}$ is smallest in the direction $\nabla\phi$, and the wavefront propagates with the velocity,

$$\frac{|\Delta\mathbf{r}|}{\Delta t} = \frac{\omega}{|\nabla\phi(\mathbf{r})|} \; . \tag{5.102}$$

which is the *phase velocity*. The phase velocity can vary from point to point in space.

Example 22 (*Phase velocity of a superposition of two plane waves*): The superposition of two plane waves with wavevectors $\mathbf{k}_{1,2} = k\hat{\mathbf{e}}_z \cos\theta \pm k\hat{\mathbf{e}}_x \sin\theta$ is described by,

$$\psi(\mathbf{r}, t) = A_0 e^{\imath(\mathbf{k}_1\cdot\mathbf{r}-\omega t)} + A_0 e^{\imath(\mathbf{k}_2\cdot\mathbf{r}-\omega t)} = 2A_0 \cos(kx\sin\theta)e^{\imath(kz\cos\theta-\omega t)} \; . \tag{5.103}$$

The phase front of this wave is a plane with normal vectors pointing along the z-axis, as illustrated in Fig. 5.16(b), and the phase velocity is now $\omega/k\cos\theta = c/\cos\theta > c$.

5.3.5.1 Beam formation by superposition of plane waves

Plane waves extend throughout the space and are uniform in transverse direction, whereas an *optical beam* is confined in transverse direction. However, as we saw in the last example, by superposing two plane waves, a resulting wave can be obtained which varies sinusoidally in transverse direction. By extrapolating this concept to superpositions of many plane waves, it is possible to construct by interference arbitrary transverse amplitude distributions. The propagation of a confined wave is the essence of *diffraction theory*. A particular case is the Gaussian beam. For mathematical simplicity and ease of visualization let us restrict ourselves to waves in two dimensions in the x-z plane. Only in the final phase will we present the complete results for three-dimensional Gaussian beams.

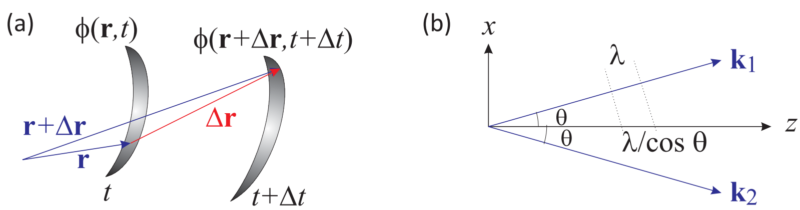

Figure 5.16: Superposition of two plane waves. The phase velocity along the direction z is higher than c, the speed of light, because in one period, the wavefront of each partial wave propagates over a distance λ, but along the z axis over a distance of $\lambda/\cos\theta$.

Before going into detailed calculations, we consider the last example again. The transverse standing wave resulting from the superposition of two plane waves, each one propagating at an angle θ with respect to the z-axis has a spatial frequency $k\sin\theta \simeq k\theta$ for small θ. We now come to a very important property of wave diffraction. Suppose that, in order to confine the wave in transverse direction, we continue adding plane waves, each one propagating at a different small angle θ, so that the amplitude adds constructively within the range $|x| < \Delta x$ and destructively out of it. By the uncertainty principle that results from the Fourier analysis and applies to this case,

$$\Delta(k\theta)\Delta x \gtrsim 1 \ . \tag{5.104}$$

That is, to confine a beam inside a width of Δx, it requires a distribution of plane waves in an angular spreading of at least $\lambda/2\pi\Delta x$. The angular spreading means that the beam will eventually diverge with an angle $\Delta\theta$.

5.3.5.2 Fresnel integrals and beam propagation

Let us now superpose plane waves in a way to form a beam. Each partial wave propagates under some angle θ with respect to the z-axis and has an amplitude

$A(\theta)d\theta$, so that the resulting wave (omitting the harmonic temporal variation) is,

$$\psi(x,z) = \int d\theta A(\theta) e^{\imath kx\sin\theta + \imath kz\cos\theta} \ . \tag{5.105}$$

In the so-called *paraxial approximation*, $A(\theta)$ is significant only within a small angular interval close to zero. This means that, according to Eq. (5.104), the transverse dimension of the beam is large in comparison to the wavelength. Expanding the trigonometric functions up the order θ^2,

$$\psi(x,z) \simeq \int d\theta A(\theta) e^{\imath kx\theta + \imath kz(1-\theta^2/2)} = e^{\imath kz} \int d\theta A(\theta) e^{\imath kx\theta - ikz\theta^2/2} \ . \tag{5.106}$$

This wave can be considered as a plane wave, $e^{\imath kz}$, modulated by the integral of (5.106). The expression (5.106) completely describes the propagation of the wave, provided that the wave is known at some point, say $z = 0$. In fact, at $z = 0$, the expression (5.106) for $\psi_0(x) \equiv \psi(x, 0)$ is a Fourier transform, whose inverse yields the angular distribution,

$$A(\theta) = \frac{k}{2\pi} \int d\xi \psi_0(\xi) e^{-\imath k\xi\theta} \ . \tag{5.107}$$

Substitution of $A(\theta)$ back into Eq. (5.106) gives,

$$\psi_z(x) \equiv \psi(x,z) = \frac{k}{2\pi} e^{\imath kz} \int d\theta \int d\xi \psi_0(\xi) e^{\imath(k\theta x - k\theta\xi - kz\theta^2/2)} \ . \tag{5.108}$$

From here on, in order to emphasize the different roles played by the transverse coordinates x and y, we will label the axial position z as an index to the wave function.

We can first integrate over θ via a quadratic extension of the exponent. The result,

$$\frac{k}{2\pi} e^{\imath kz} \int d\theta e^{\imath(k\theta x - k\theta\xi - kz\theta^2/2)} = \sqrt{\frac{k}{2\pi\imath z}} e^{\imath k(z+(x-\xi)^2/2z} \equiv h_z(x-\xi) \ , \tag{5.109}$$

gives us the field at the position z as an integral over ξ of the field in $z = 0$, $\psi_0(\xi)$. The expression (5.109) is called *impulse response*, *kernel*, *propagator*, or *Green's function*, depending on the context. Carry out the integral (5.109) in Exc. 5.3.6.17.

The kernel has very simple physical interpretations: It is the field at point (x, z) generated by a point source with unitary amplitude located in $(\xi, 0)$. In the same time, it is a (two-dimensional) spherical wave in a paraxial form. To see this, we write the field of a two-dimensional spherical wave (i.e. a circular wave) with its center in $(\xi, 0)$ as,

$$\sqrt{\frac{1}{r}} e^{\imath kr} \ , \tag{5.110}$$

where $r = \sqrt{(x-\xi)^2 + z^2}$. (Instead of $1/r$ as in three dimensions, the amplitude decreases as $\sqrt{1/r}$ in two dimensions.) Near the z-axis, we approximate $r \simeq z + (x - \xi)^2/2z$, and the spherical wave becomes,

$$\sqrt{\frac{1}{z}} e^{\imath k[z+(x-\xi)^2/2z]} \ , \tag{5.111}$$

which is the same expression as $h_z(x-\xi)$ in Eq. (5.109). Note that the quadratic term in $x-\xi$ can become considerable in comparison with the wavelength. Eq. (5.106) now becomes,

$$\boxed{\psi_z(x) = \int h_z(x-\xi)\psi_0(\xi)d\xi = \sqrt{\frac{k}{2\pi\imath z}}e^{\imath kz}\int e^{\imath k(x-\xi)^2/2z}\psi_0(\xi)d\xi} \ . \tag{5.112}$$

We will call this integral the *Fresnel integral*. It is the mathematical expression of the *Huygens principle*: The field in (x,z) is the sum of all spherical waves centered on all previous points $(\xi,0)$ weighed with the respective field amplitude $\psi_0(\xi)$ [2].

The expressions (5.106) and (5.112) represent two equivalent ways to calculate wave propagation. Eq. (5.106) calculates the wave from the angular distribution of its plane wave components. When the angular distribution is of Hermite-Gaussian type, a Gaussian beam results. In contrast, Eq. (5.112) computes the wave at a point z from the field at an initial point $z=0$. This is the traditional theory of Fresnel diffraction. Here, also a Gaussian beam results when ψ_0 is Hermite-Gaussian.

To deepen our understanding of beam propagation let us introduce the important concept of *near field* and *far field*. By 'near field' we mean a distance z sufficiently small to be allowed to neglect the quadratic term in the exponent of Eq. (5.106),

$$k\theta^2 z \ll 1 \ . \tag{5.113}$$

Then the near field, in zero-order approximation, is precisely the field at $z=0$ multiplied with propagation phase factor $e^{\imath kz}$,

$$\psi_z(x) \simeq e^{\imath kz}\int d\theta A(\theta)e^{\imath kz\theta} = e^{\imath kz}\psi_0(x) \ , \tag{5.114}$$

where the second equation follows from Eq. (5.107). Let us now examine the first-order correction and define 'near' more precisely.

The question is, what is the maximum angle of θ allowed in (5.113)? It is not $\pi/2$, but rather, it is the range of angles over which $A(\theta)$ is significantly different from zero. This angular range $\Delta\theta$ is related to the range of transverse distance Δx via the Fourier transform (5.104), so that,

$$\frac{\pi\Delta x^2}{\lambda} \gg z/2 \ . \tag{5.115}$$

The quantity on the left side, called the *Rayleigh range*, is the demarcation between the near and far field regimes. A simple physical interpretation for this quantity will be given below.

Let us now investigate the 'far field' regime of large z having a closer look at Eq. (5.112). When ψ_0 is confined to Δx, and if z is sufficiently large for the quadratic factor to be,

$$k\xi^2/2z \ll 1 \ , \tag{5.116}$$

or

$$\frac{\pi\Delta x^2}{\lambda} \ll z \ , \tag{5.117}$$

then it can be ignored, and the integral becomes,

$$\psi(x,z) \simeq \sqrt{\frac{k}{\imath 2\pi z}} e^{\imath k(z+x^2/2z)} \int e^{-\imath kx\xi/z}\psi_0(\xi)d\xi \ . \tag{5.118}$$

We see that the amplitude of the far field is given by the amplitude of the Fourier transform of the field at $z = 0$ except for a quadratic phase factor $kx^2/(2z)$ [5]

Let us go back to the near field and calculate the first-order correction. For small $z = \Delta z$, we can expand the exponent in equation (5.106),

$$\begin{aligned} \psi_z(x) &\simeq e^{\imath k\Delta z} \int d\theta A(\theta) \left(1 - \imath k\frac{\theta^2}{2}\Delta z\right) e^{\imath k\theta x} \\ &= e^{\imath k\Delta z}\psi_0(x) - e^{\imath k\Delta z}\frac{\imath k\Delta z}{2} \int d\theta A(\theta)\theta^2 e^{\imath k\theta x} \ . \end{aligned} \tag{5.119}$$

The last integral is,

$$\int d\theta A(\theta)\theta^2 e^{\imath k\theta x} = -\frac{1}{k^2}\frac{\partial^2}{\partial x^2}\int d\theta A(\theta) e^{\imath k\theta x} = -\frac{1}{k^2}\frac{\partial^2\psi_0(x)}{\partial x^2} \ , \tag{5.120}$$

Such that close to $z = 0$, we get,

$$\psi_z(x) \simeq e^{\imath k\Delta z}\left[\psi(x,0) + \frac{\imath\Delta z}{2k}\frac{\partial^2\psi(x,0)}{\partial x^2}\right] \ . \tag{5.121}$$

Note that the first-order correction is in quadrature with the zero-order term (5.114) (if ψ_0 is real), which means that the correction is in the phase, not in the amplitude. The second derivative can be seen as a diffusion operator [6], and it is this phase diffusion, which is the cause of phenomenon of *diffraction*.

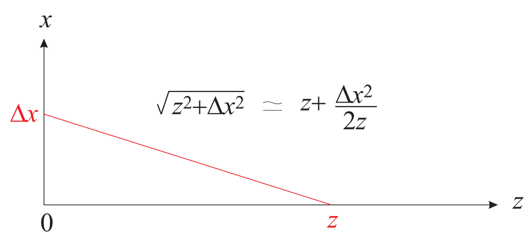

Figure 5.17: Illustration of the Rayleigh range: The distance from $(0,0)$ to $(0,z)$ is z. The distance from $(0,\Delta x)$ to $(0,z)$ is approximately $z + \Delta x^2/(2z)$. The difference is $\Delta x^2/(2z)$. Thus, a wave coming from $(0,0)$ and a wave coming from $(0,\Delta x)$ will acquire a phase difference of $k\Delta x^2/(2z)$ when they reach $(0,z)$. The phase difference is equal to 1 when z equals the Rayleigh range. The phase difference is insignificant in the far field, but significant in the near field.

We can generalize a little more: Suppose we write ψ as a plane wave $e^{\imath kz}$ modulated by a function with slow variation $u(x,z)$,

$$\psi_z(x) \equiv u_z(x)e^{\imath kz} \ , \tag{5.122}$$

[5] In fact, the phase factor can be circumvented by choosing z equal to a focal length f of a lens.

[6] This is because the second derivative of a Gaussian function is negative in the center and positive in the wings, so that when added to the original function, the distribution is reduced in the center and increased in the wings.

then

$$u_{\Delta z}(x) - u_0(x) = \frac{\imath\Delta z}{2k}\frac{\partial^2 u_0(x)}{\partial x^2} \ . \tag{5.123}$$

We derived this relation for a particular point on the z-axis, $z = 0$. However, there is no particular need to choose this point, and the relationship applies to any z. Thus, letting $\Delta z \to 0$, we get,

$$2ik\frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial x^2} = 0 \ . \tag{5.124}$$

This equation is called *paraxial wave equation*. It is an approximate form of the scalar wave equation and has the same form as the Schrödinger equation for a free particle. The equation can be generalized to three dimensions by a similar derivation:

$$\boxed{2\imath k\frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \ . \tag{5.125}$$

The *Fresnel integral is the solution of the paraxial wave equation* with a boundary condition for ψ at $z = 0$. We will show in Sec. ?? that a three-dimensional wave can be constructed from two-dimensional waves [5]. The resulting Fresnel integral in three dimensions is,

$$\boxed{\psi_z(x,y) = \frac{e^{\imath kz}}{\imath\lambda z}\int e^{\imath k(x-\xi)^2/2z}e^{\imath k(y-\eta)^2/2z}\psi_0(\xi,\eta)d\xi d\eta} \ , \tag{5.126}$$

where $\psi_0(x,y)$ is the distribution of the field amplitude at $z = 0$. Note that, as required by energy conservation, in three dimensions the field decays like $1/z$ and not like $\sqrt{1/z}$, as it does in two dimensions. Note also that the pulse response in three dimensions is essentially the product of two two-dimensional pulse responses.

5.3.5.3 Application of Fresnel diffraction theory

The Fresnel diffraction integral, Eq. (5.112), can be applied in various situations illustrating its use and the difference between wave optics and geometric optics. Examples are the diffraction through a slit, the pin-hole camera, the focusing of a thin lens, etc. [5].

Near-field diffraction (also called *Fresnel diffraction*) and far-field diffraction (also called *Fraunhofer diffraction*) are often distinguished by a quantity called the *Fresnel number*,

$$F \equiv \frac{a^2}{z\lambda} \ , \tag{5.127}$$

where a is the size of the beam (or *aperture*). The near field zone is defined by $F \gtrsim 1$, whereas in the far field zone, $F \ll 1$. For a Gaussian beam, letting $a = \sqrt{\pi}w_0$, we recover the Rayleigh length condition for Fresnel diffraction $z \lesssim z_R$, respectively Fraunhofer diffraction, $z \gg z_R$.

5.3.6 Exercises

5.3.6.1 Ex: Waves on a rope

A string with linear mass density μ is attached at two points distant by $L = 1\,\text{m}$. A mass of $m = 1\,\text{kg}$ is attached to one end of the string that goes over a pulley, as shown in the figure. Excited by a vibrating pin with frequency $f = 1\,\text{kHz}$ the string performs transverse vibrations with the wavelength $\lambda = 2L$.
a. Calculate the sound velocity.
b. Now the mass is replaced by a mass $m' = 4m$. Calculate the new sound velocity.
c. Assuming the sound velocity, how often should the pin excite the string to observe the third oscillation mode (three anti-nodes)?

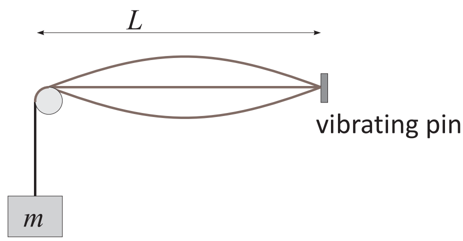

Figure 5.18: Waves on a rope.

5.3.6.2 Ex: Optical cavity

Optical cavities consist of two light reflecting mirrors. Standing light waves must satisfy the condition that the electric and magnetic fields vanish on the mirror surfaces. What is the frequency difference between two consecutive modes of a of length $L = 10\,\text{cm}$?

5.3.6.3 Ex: Waves on a rope

A string vibrates according to the equation $y(x,t) = 15 \sin \frac{\pi x}{4\,\text{cm}} \cos(30\,\text{s}^{-1}\,\pi t)$.
a. What is the velocity of a string element at the position $x = 2\,\text{cm}$ at the instant of time $t = 2\,\text{s}$?
b. What is the propagation speed of this wave?

5.3.6.4 Ex: Violin

The length of a violin string is $L = 50\,\text{cm}$, and its mass is $m = 2.0\,\text{g}$. When it is attached at the ends, the string can emit the a'-pitch ('la') corresponding to 440 Hz. Where should a finger be placed so that the emitted sound is the c"-pitch ('do') at 528 Hz?

5.3.6.5 Ex: Sound waves

The air column inside a closed tube, filled with a gas whose characteristic sound velocity is v_s, is excited by a speaker vibrating at the frequency f. Gradually increasing

the frequency of the speaker one observes that the tube emits a sound at $f = 440\,\mathrm{Hz}$ and the next time at $660\,\mathrm{Hz}$.
a. What is the length of the tube?
b. What is the speed of sound?

5.3.6.6 Ex: Sound in a bottle

An experimenter blows into a bottle partially filled with water producing a sound of $1000\,\mathrm{Hz}$. After drinking some of the water until the level decreased by 5 cm he is able to produce a sound at $630\,\mathrm{Hz}$. Determine the possible values for the speed of sound knowing that the vibration of the air column inside the bottle should have a node at the end which is in contact with water and an anti-node at the mouth of the bottle. Comparing the result to the known value for the speed of sound in air, what is the excited vibration mode?

5.3.6.7 Ex: Sonic waves in a tube

The figure shows a rod fixed at its center to a vibrator. A disc attached to the end of the rod penetrates a glass tube filled with a gas and where cork dust had been deposited. At the other end of the tube there is a movable piston. When producing longitudinal vibrations at the rod, we note that for certain positions of the movable piston, the cork dust forms a pattern of node and anti-nodes. Knowing for one of the positions of the piston the distance d between the anti-nodes and the frequency f of the vibration, show that the speed of sound in the gas is $v = 2fd$. This is called Kundt's method for determining the speed of sound in a gas.

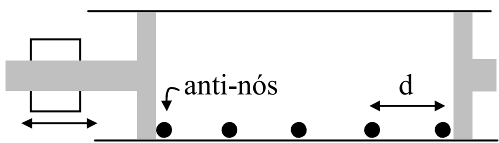

Figure 5.19: Sonic waves in a tube.

5.3.6.8 Ex: Sound filter

A tube can act as an acoustic filter discriminating various sound frequencies crossing it from its own frequencies. A car muffler is an application example.
a. Explain how this filter works.
b. Determine the 'cut-off' frequency below which sound is not transmitted.

5.3.6.9 Ex: Snell's law

Derive Snell's law from Huygens principle.

5.3.6.10 Ex: Surface gravitational waves, capillary waves

Dependence of the propagation velocity on the height of the water column.

5.3.6.11 Ex: Propagating standing wave

Consider two propagating waves $\mathcal{E}_\pm(x,t)$ with equal amplitudes and slightly different frequencies $\omega_\pm$ propagating in opposite directions along the x-axis.
a. Show that, approximating $k_+ \simeq k_-$, at each instant of time the interference pattern along the x-axis forms a standing wave.
b. Determine the group velocity of this wave.

5.3.6.12 Ex: Mach-Zehnder and Michelson-interferometer

Interferometers are devices that allow the comparison of distances via the propagation time of waves taking different paths. The interferometers outlined in the figures are based on beam splitters that divide and recombine a wave described by $I_n(x,t) = A_n \cos(kx - \omega t)$. Determine the amplitude of the signal at the position of the beam splitter recombining the waves as a function of a variation $\Delta x = 4\pi/k$ of the length of the second interferometer arm.

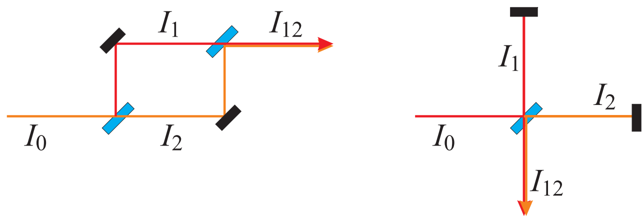

Figure 5.20: Mach-Zehnder and Michelson-interferometer.

5.3.6.13 Ex: Multiple interference in optical cavities

An optical beam splitter is a mirror with partial transmission and partial reflection,

$$\mathcal{E}_r(x,t) = \pm r\mathcal{E}_0(x,t) \quad , \quad \mathcal{E}_t(x,t) = t\mathcal{E}_0(x,t) \ .$$

The reflection signal depends on the direction of incidence, because reflection at a denser medium introduces a phase shift of π. Using this rules derive for a set of two mirrors r_1 and r_2 separated by a distance L the field $\mathcal{E}_{cav}$ between the mirrors as a function of the wave vector of the incident field $\mathcal{E}_{in}$. Also calculate the amplitudes of the transmitted and reflected light. Calculate the phase shifts between the transmitted (reflected) light and the incident light. Interpret the results.

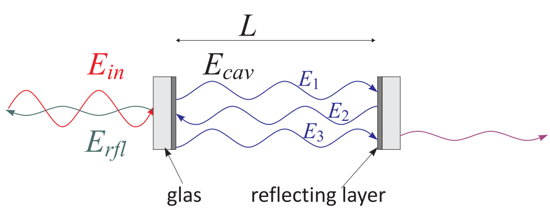

Figure 5.21: Optical cavity.

5.3.6.14 Ex: Double slit

Calculate the interference pattern behind a double slit.

5.3.6.15 Ex: Spherical waves

Show that spherical waves given by $Y(\mathbf{r},t) = \frac{Y_0}{kr}\sin(kr-\omega t)$ satisfy the 3D wave equation. Use Cartesian coordinates.

5.3.6.16 Ex: Interference in spherical waves

Two spherical waves are generated at positions $\mathbf{r}_\pm = \pm R\hat{\mathbf{e}}_z$. Determine surfaces of destructive interference for these waves.

5.3.6.17 Ex: Green's function

Calculate the integral Eq. (5.109).

5.4 Fourier analysis

Every periodic function $f(\xi) = f(\xi+2\pi)$ can be decomposed into a series of harmonic vibrations. This is the *Fourier theorem*,

$$\boxed{f(\xi) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\xi + b_n\sin n\xi)}\ . \tag{5.128}$$

To determine the coefficients, we calculate,

$$\int_0^{2\pi} f(\xi)d\xi = \int_0^{2\pi}\left[\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m\cos m\xi + b_m\sin m\xi\right]d\xi = \pi a_0 \tag{5.129}$$

$$\int_0^{2\pi} f(\xi)\cos k\xi d\xi = \int_0^{2\pi}\left[\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m\cos m\xi + b_m\sin m\xi\right]\cos n\xi d\xi = \pi a_n$$

$$\int_0^{2\pi} f(\xi)\sin k\xi d\xi = \int_0^{2\pi}\left[\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m\cos m\xi + b_m\sin m\xi\right]\sin n\xi d\xi = \pi b_n\ ,$$

using the rules,

$$\int_0^{2\pi}\cos n\xi\cos m\xi d\xi = \int_0^{2\pi}\sin n\xi\sin m\xi d\xi = \pi\delta_{n,m} \quad \text{and} \quad \int_0^{2\pi}\cos n\xi\sin m\xi dx = 0\ . \tag{5.130}$$

We can use these equations to calculate the Fourier expansion. To simplify the calculations, it is useful to consider the symmetry of the periodic function, since if $f(\xi) = f(-\xi)$, we can neglect all the coefficients b_n, and if $f(\xi) = -f(-\xi)$, we can neglect the coefficients b_n [7]

[7] Alternatively we can write the theorem as,

$$f(\xi) = \sum_{n=-\infty}^{\infty} d_n e^{\imath n\xi}\ ,$$

Example 23 (*Frequency spectrum and low-pass filter*):

- Show the spectrum of a rectangular signal on an oscilloscope and on an spectrum analyzer.
- Show the same spectrum filtered by a low pass filter.

.

5.4.1 Expansion of vibrations

Interpreting $\xi \equiv \omega t$ as time, we can apply the Fourier theorem (5.128) on temporal signals, $S(t) = f(\omega t)$, where ω is the angular frequency,

$$S(t) = f(\omega t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t) \ , \tag{5.131}$$

with,

$$a_0 = \frac{\omega}{\pi}\int_0^{2\pi/\omega} S(t)dt \quad \text{and} \quad a_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} S(t)\cos n\omega t dt \tag{5.132}$$
$$\text{and} \quad b_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} S(t)\sin n\omega t dt \ .$$

The representation of the coefficients a_n and b_n as functions of the number n is called *harmonic spectrum*. As we mentioned earlier, the spectrum of a sound is what determines the timbre. The total *harmonic distortion* is defined by,

$$k \equiv \frac{\sum_{n=2}^{\infty}(a_n + b_n)}{\sum_{n=1}^{\infty}(a_n + b_n)} \ . \tag{5.133}$$

Radiofrequency circuits such as HiFi amplifiers are characterized by their transmission fidelity, that is, the absence of harmonic distortion in the amplification of each harmonic coefficient.

5.4.1.1 Expansion of a triangular signal

We consider a *triangular signal* given by [8],

$$S(t) = \begin{cases} \omega t & \text{for} \quad 0 < \omega t < \frac{\pi}{2} \\ \pi - \omega t & \quad\quad\;\; \frac{\pi}{2} < \omega t < \pi \end{cases} \ . \tag{5.134}$$

---

determining the coefficients as,

$$\int_{-\pi}^{\pi} f(\xi)e^{-\imath k\xi}d\xi = \int_{-\pi}^{\pi} f(\xi)e^{-\imath k\xi}d\xi \sum_{n=-\infty}^{\infty} d_n e^{\imath n\xi}d\xi = 2\pi d_n \ ,$$

with,

$$2d_n = a_n - \imath b_n \quad \text{for} \quad n \geq 0 \quad \text{and} \quad 2d_n = a_{-n} + \imath b_{-n} \quad \text{for} \quad n < 0 \ .$$

[8]Note that the function $S(t) = \frac{\pi}{2} - \left(\frac{\pi}{2} - \omega t\right)\frac{\cos \omega t}{|\cos \omega t|}$, which describes the same triangular signal, it is easier to program in numerical softwares.

We calculate the coefficients, $a_0 = 0$, because the signal is symmetric about the t-axis (it has no offset), and $a_n = 0$, because the signal has the symmetry $S(t) = -S(-t)$. Also,

$$b_n = \frac{2\omega}{\pi}\int_0^{\pi/2\omega} \omega t \sin n\omega t dt + \frac{2\omega}{\pi}\int_{\pi/2\omega}^{\pi/\omega} (\pi - \omega t) \sin n\omega t dt = \frac{4}{\pi}\frac{\sin\frac{1}{2}\pi n}{n^2} \;, \tag{5.135}$$

with the consequence,

$$S(t) = \frac{4}{\pi}\sum_{n=1,3,\ldots} \frac{(-1)^{(n-1)/2}}{n^2} \sin n\omega t \;. \tag{5.136}$$

5.4.2 Theory of harmony

Non-linearities in oscillating systems can excite harmonic frequencies f_n, that is, multiples of the fundamental frequency $f_n = (n+1)f$. These are the components of the Fourier series.

All musical instruments produce harmonics. This is what makes the timbre of the instrument. When we play several notes together, we perceive the octave interval as pleasant. This is, because all the harmonics of a pitch and of its octave coincide.

- harmonic pitch, well-tempered chromatic scale, flat $\flat$, sharp $\sharp$, $\natural$, musical clef, tuning fork $f_{a'} = 440\,\text{Hz}$

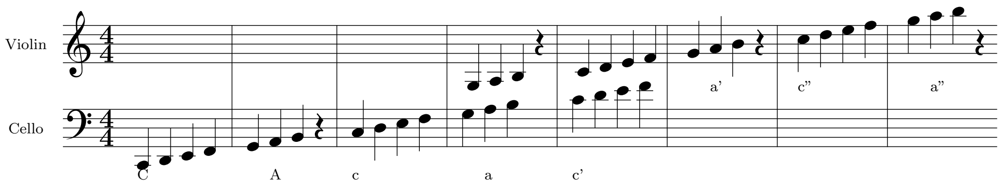

Figure 5.22: (code) Ladder of pitches over 3 octaves. Pitches can be generated in MATLAB. A sample program can be downloaded by clicking on the link.

In the *well-tempered* tonality the interval of a *octave* is divided into 12 intervals,

$$n \in [\text{a, a\#, h, c, c\#, d, d\#, e, f, f\#, g, g\#}] \;. \tag{5.137}$$

Defining *normal tuning* as,

$$f_a = 440\,\text{Hz} \;, \tag{5.138}$$

the pitches correspond to the frequencies,

$$f_n = 2^{n/12} f_a \;. \tag{5.139}$$

For example, we calculate the frequency of the 'd',

$$f_d = 2^{-7/12} f_a = 391.9954\,\text{Hz} \;. \tag{5.140}$$

Thus, all notes are logarithmically equidistant:

$$\text{lb } f_{n+1} - \text{lb } f_n = \text{lb } (2^{(n+1)/12} f_{la}) - \text{lb } (2^{n/12} f_{la}) = 1 \tag{5.141}$$

$$\frac{f_{n+1}}{f_n} = \frac{2^{(n+1)/12} f_{la}}{2^{n/12} f_{la}} = 2^{1/12} \ . \tag{5.142}$$

Why are there just 12 pitches? Several instruments have more than one resonator emitting sound, e.g. the violoncello has 4 strings, c, g, d', and a'. Each string is detuned by a *quint* from the next string, that is,

$$3f_c = 2f_g \quad \text{and} \quad 3f_g = 2f_{d'} \quad \text{and} \quad 3f_{d'} = 2f_{a'} \ . \tag{5.143}$$

Each string has its own series of harmonics. The timbre of the instrument appears more pleasant, when the harmonics of the various strings coincide. Let us now check, whether our definition of logarithmically equidistant pitches satisfies this condition,

$$3f_d = 3 \cdot 2^{-7/12} f_a = 2.0023 f_a \neq 2f_a \ . \tag{5.144}$$

Thus, harmonic tuning is not perfect, but quite close to the well-tempered tuning. In Exc. 5.4.6.5 we show that, nevertheless, the discrepancy is able to produce nasty beat notes.

The guitar, which is tuned in *quart*s,

$$4f_c = 3f_{fa} \ , \tag{5.145}$$

has the same problem [9],

$$4f_c = 4 \cdot 2^{-5/12} f_f = 2.9966 f_f \neq 3f_f \ . \tag{5.146}$$

Resolve the Excs. 5.4.6.6, 5.4.6.7 and 5.4.6.8.

5.4.3 Expansion of waves

Interpreting $\xi \equiv kx$ as position, we can apply the Fourier theorem (5.128) to standing waves, $Y(x) = f(kx)$, where $k = 2\pi/k$ is the wavevector.

5.4.4 Normal modes in continuous systems at the example of a string

We will now apply the Fourier expansion to calculate the normal modes of a vibrating string. Depending on which mode of oscillation is excited, the displacement of the string is given by,

$$Y_n(x,t) = (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin \tfrac{\omega_n x}{c} \ , \tag{5.147}$$

where $\omega_n = n\pi c/l$ is the frequency of the *normal mode*. An arbitrary vibration can be decomposed as superpositions of these modes,

$$Y(x,t) = \sum_n Y_n(x,t) \ . \tag{5.148}$$

[9] Include Matlab sound examples here!

As an initial condition we assume that the string is at a position $Y(x,0) = Y_0(x)$ with the velocity $V(x,0) = V_0(x)$ at all points. Then,

$$Y_0(x) = \sum_n Y_n(x,0) = \sum_n A_n \sin \tfrac{\omega_n x}{c} \tag{5.149}$$

$$\text{and} \qquad V_0(x) = \sum_n \frac{d}{dt} Y_n(x,0) = \sum_n \omega_n B_n \sin \tfrac{\omega_n x}{c} \ .$$

We find the amplitudes by calculating the integrals,

$$\frac{2}{l}\int_0^l Y_0(x) \sin \tfrac{\omega_n x}{c} dx = \frac{2}{l}\sum_m A_m \int_0^l \sin \tfrac{m\pi x}{l} \sin \tfrac{n\pi x}{l} dx = A_n \tag{5.150}$$

$$\frac{2}{l}\int_0^l V_0(x) \sin \tfrac{\omega_n x}{c} dx = \frac{2}{l}\int_0^l \sum_m \omega_n B_m \sin \tfrac{m\pi x}{l} \sin \tfrac{n\pi x}{l} dx = \omega_n B_n \ .$$

We now assume that the rope is initially excited by a triangular deformation, that is, we pull the rope in its middle up to a distance d and let go. That is, the initial conditions are given by,

$$V_0(x) = 0 \qquad \text{and} \qquad \frac{\pi}{2d} Y_0(x) = \begin{cases} \frac{\pi x}{l} & \text{for} \quad 0 < \frac{\pi x}{l} < \frac{\pi}{2} \\ \frac{\pi(l-x)}{l} & \phantom{\text{for}} \quad \frac{\pi}{2} < \frac{\pi x}{l} < \pi \end{cases} \ . \tag{5.151}$$

We can compare this function with the triangle function Eq. (5.134) and make the same Fourier expansion as in (5.136),

$$\frac{\pi}{2d} Y_0(x) = \frac{4}{\pi} \sum_{n=1,3,\ldots} \frac{(-1)^{(n-1)/2}}{n^2} \sin \tfrac{n\pi x}{l} \ . \tag{5.152}$$

Comparing this expansion with (5.149), we find $B_n = 0$ and,

$$\sum_m A_m \sin \tfrac{\omega_m x}{c} = Y_0(x) = \frac{2d}{\pi}\frac{4}{\pi} \sum_{n=1,3,\ldots} \frac{(-1)^{(n-1)/2}}{n^2} \sin \tfrac{n\pi x}{l} \ . \tag{5.153}$$

yielding for odd coefficients $m = 1, 3, ..,$

$$A_n = \frac{8d}{n^2\pi^2}(-1)^{(n-1)/2} \ . \tag{5.154}$$

Thus, the vibration of the string is completely described by,

$$Y(x,t) = \frac{8d}{\pi^2} \sum_{n=1,3,\ldots} \frac{(-1)^{(n-1)/2}}{n^2} \cos \omega_n t \sin \tfrac{\omega_n x}{c} \ . \tag{5.155}$$

The energy is the sum of the energies of all normal modes,

$$E = \sum_{n=1,3,\ldots} \frac{m}{4}\omega_n^2 A_n^2 = \sum_{n=1,3,\ldots} \tfrac{m}{4}\left(\frac{n\pi c}{l}\right)^2 \left(\frac{8d}{n^2\pi^2}\right)^2 = \sum_{n=1,3,\ldots} m\frac{16d^2c^2}{n^2\pi^2 l^2} = \frac{2md^2c^2}{l^2} \ , \tag{5.156}$$

knowing $\sum_{n=1,3,\ldots} \frac{1}{n^2} = \frac{\pi^2}{8}$.

5.4.5 Waves in crystalline lattices

The sound may propagate in a crystalline lattice, for example a metal or a crystal, by means of longitudinal or transverse vibrations. To understand the propagation of longitudinal vibrations in a monoatomic lattice, we consider the model of a chain of N masses coupled by springs. The treatment for transverse vibrations is analogous. As we have shown in previous sections, the movement of each mass is described by the differential equation,

$$\ddot{x}_n = \omega_0^2(x_n - x_{n-1}) + \omega_0^2(x_n - x_{n+1}) \ , \tag{5.157}$$

with $n = 1, .., N$. Making the ansatz $x_n = A_n e^{-\imath\omega t}$, we obtain the characteristic equation,

$$\omega^2 A_n = \omega_0^2(A_n - A_{n-1}) + \omega_0^2(A_n - A_{n+1}) \ . \tag{5.158}$$

When we hit one of the oscillators of a linear chain, we excite a wave that propagates along the chain. Therefore, it is reasonable to guess $A_n = Ae^{\imath nka}$ for the displacements of the oscillators, where $a \equiv x_{n+1} - x_n$ is the lattice constant. We obtain,

$$\omega^2 = \omega_0^2(1 - e^{-ika}) + \omega_0^2(1 - e^{\imath ka}) = 2\omega_0^2(1 - \cos ka) = 4\omega_0^2 \sin^2 \frac{ka}{2} \ . \tag{5.159}$$

The dispersion relation is shown in Fig. 5.23. Obviously, in the limit of long waves, $ka \ll 1$, the relation can be approximated by,

$$\omega = 2\omega_0 \left| \sin \frac{ka}{2} \right| \simeq \omega_0 ka \equiv ck \ , \tag{5.160}$$

where c is the propagation velocity of the wave. This relation is linear, thus reproducing the situation of acoustic waves.

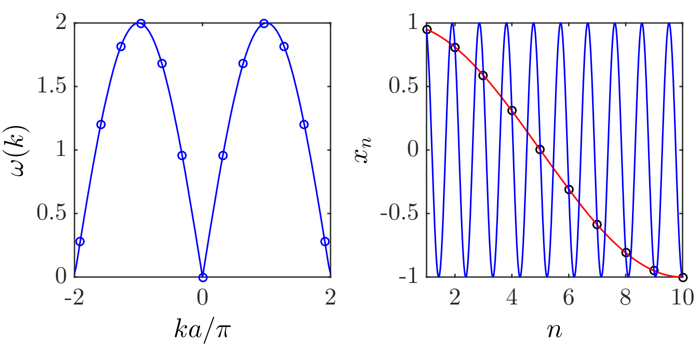

Figure 5.23: (code) Dispersion relation of a one-dimensional crystalline lattice consisting of 20 atoms.

The displacements of individual oscillators are now,

$$x_n(t) = na + Ae^{\imath nka - \imath\omega t} \ . \tag{5.161}$$

We need now to discuss, what are the possible values for k. First, since by adding 2π to the value ka we get the same result, we may concentrate on the region $-\pi < ka < \pi$, called the first *Brillouin zone*. And since the crystal is symmetric (we can reverse the order of all oscillators), we can assume cyclic boundary conditions, $e^{\imath nka} = e^{\imath(N-n)ka}$, such that $(N-2n)ka/2\pi$ is an arbitrary integer number for any n, for example $n = 0$,

$$k = \frac{2\pi}{Na} \cdot \ell \ , \tag{5.162}$$

for $\ell \in \mathbb{N}$. To stay within the Brillouin zone, we set $\ell = -\frac{N}{2}, .., \frac{N}{2}$. That is, we have N possible values, which corresponds to just half the number of degrees of freedom.

Let us consider particular solutions. In the center of the Brillouin zone, $k = 0$ we have,

$$x_n(t) = na + Ae^{\imath\omega t} \ , \tag{5.163}$$

which corresponds to an in-phase oscillation of all oscillators. On the edge of the Brillouin zone, $k = \pm\pi/a$,

$$x_n(t) = na + A(-1)^n e^{\imath\omega t} \ , \tag{5.164}$$

which corresponds to a movement, where consecutive oscillators oscillate in anti-phase.

5.4.5.1 Waves in diatomic crystalline lattices

Many lattices are diatomic, that is, made of two species of atoms with different masses. For example, the NaCl salt crystal is a lattice alternating Na^+ and Cl^- ions. In analogy with the monoatomic lattice we establish the equations of motion,

$$\begin{aligned} \ddot{x}_n &= -\omega_x^2(x_n - y_{n-1}) - \omega_x^2(x_n - y_n) \\ \ddot{y}_n &= -\omega_y^2(y_n - x_{n+1}) - \omega_y^2(y_n - x_n) \ , \end{aligned} \tag{5.165}$$

with $\omega_{x,y} \equiv \sqrt{k/m_{x,y}}$. Inserting the ansätze $x_n = Ae^{\imath(nka-\omega t)}$ and $y_n = Be^{\imath(nka-\omega t)}$, we find the equations,

$$\begin{aligned} -\omega^2 A &= -\omega_x^2(2A - Be^{-\imath ka} - B) \\ -\omega^2 B &= -\omega_y^2(2B - Ae^{\imath ka} - A) \ , \end{aligned} \tag{5.166}$$

or,

$$\begin{pmatrix} 2\omega_x^2 - \omega^2 & -\omega_x^2(1 + e^{-\imath ka}) \\ -\omega_y^2(1 + e^{\imath ka}) & 2\omega_y^2 - \omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \ . \tag{5.167}$$

The characteristic equation is,

$$0 = \det \hat{M} = (2\omega_x^2 - \omega^2)(2\omega_y^2 - \omega^2) - \omega_x^2(1 - e^{-\imath ka})\omega_y^2(1 - e^{\imath ka}) \ , \tag{5.168}$$

with the solution,

$$\omega^2 = \omega_x^2 + \omega_y^2 \pm \sqrt{\omega_x^4 + \omega_y^4 + 2\omega_x^2\omega_y^2 \cos ka} \ . \tag{5.169}$$

For $ka \ll 1$ we can approximate,

$$\omega^2 \simeq \omega_x^2 + \omega_y^2 \pm \sqrt{(\omega_x^2+\omega_y^2)^2 - \omega_x^2\omega_y^2 k^2 a^2} \tag{5.170}$$
$$\simeq 2(\omega_x^2+\omega_y^2) \quad , \quad \omega_x^2\omega_y^2 k^2 a^2 \ .$$

The first eigenvalue is called the *optical branch* and the second the *acoustic branch*. The optical branch corresponds to an anti-phase motion of the atoms of the species x and y. This motion can be excited by light fields. The acoustic branch corresponds to an in-phase motion of the atoms.

In contrast, for $ka \simeq \pm\pi/a$ we obtain,

$$\omega^2 = \omega_x^2 \quad , \quad \omega_y^2 \ . \tag{5.171}$$

In these solutions either atom x oscillates while y stays at rest, or the opposite.

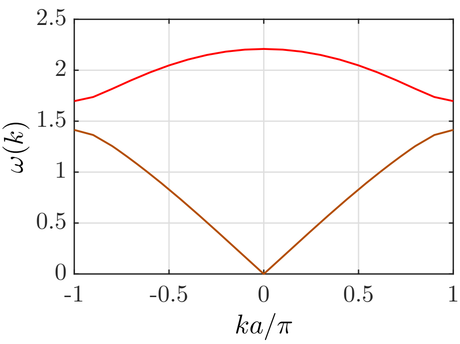

Figure 5.24: (code) Dispersion relation in a one-dimensional lattice showing in blue the optical branch and in green the acoustic branch.

5.4.6 Exercises

5.4.6.1 Ex: Fourier expansion

Expand the function $f(\xi) = \sin^3 \xi$ in a Fourier series.

5.4.6.2 Ex: Fourier expansion of sea waves

Surface waves on the sea are often better described by the function $f(x,t) = (kx - 2n\pi)^2$ inside the intervals $x \in [(2n-1)\pi/k, (2n+1)\pi/k]$ com $n \in \mathbb{N}$. Expands the wave in a spatial Fourier series. Use the formula $\int z^2 \cos(bz)dz = \frac{1}{b^3}[(b^2z^2-2)\sin bz + 2bz\cos bz]$.

5.4.6.3 Ex: Fourier expansion of a rectified signal

An alternating electric current can be turned into a signal of half-cycles, $f(t) = |\cos\frac{\omega t}{2}|$, by a diode rectifier bridge. Expand this signal into a temporal Fourier series. Use the formula $\int \cos(az)\cos(bz)dz = \frac{\sin[(a-b)z]}{2(a-b)} + \frac{\sin[(a+b)z]}{2(a+b)}$.

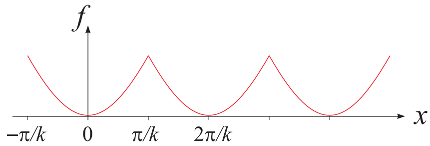

Figure 5.25: (code)

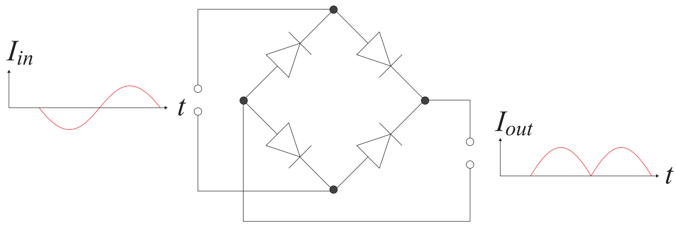

Figure 5.26: Fourier expansion of a rectified signal.

5.4.6.4 Ex: Action of a low pass filter on a spectrum

One method of creating a sinusoidal signal in electronics consist in first creating a rectangular signal via a switching circuit and then pass this signal through a low-pass filter by cutting off the harmonics. Simulate this procedure using the Fourier transform method starting from the rectangular signal $S(t) = \sin \omega t / |\sin \omega t|$ with $\omega/2\pi = 1\,\text{kHz}$ and using a low pass filter, such as $F(\omega) = 1/\left(1 + (\omega/\omega_g)^2\right)$, where the cut-off frequency is, $\omega_g/2\pi = 1\,\text{kHz}$. Evaluate the harmonic distortion of the rectangular signal and the filtered signal.

5.4.6.5 Ex: Tuning a violin

What would be the beat frequency between the pitches $3f_{c'}$ and $2f_{g'}$ if the strings were tuned logarithmically equidistant.

5.4.6.6 Ex: String instruments

Imagine a string instrument with 12 strings tuned in quints. How far would be the highest string from a harmonic of the lowest one.

5.4.6.7 Ex: String instruments

Prepare a list comparing the harmonics up to ninth order in the harmonic and in the tempered scale.

5.4.6.8 Ex: Frequency beating of sound waves

To tune a violin a musician first tunes the a-string ('la') at $f_a = 440\,\text{Hz}$ and then plays two neighboring strings, paying attention to the frequency beats. When playing the a- and the e-string ('mi'), the violinist hears a beat frequency of $3\,\text{Hz}$, and he notes that this frequency increases as the tension of the e-string increases. (The e-string is

tuned to $f_e = 660\,\text{Hz}$.)
a. Why is there a beat when the two strings are played simultaneously?
b. What is the vibration frequency of the e-string when the beat frequency it generates together with the a-string is $3\,\text{Hz}$?
c. If the tension on the e-string is $80\,\text{N}$ for a beat frequency of $3\,\text{Hz}$, what tension corresponds to a perfect tuning of the string?

5.4.6.9 Ex: Frequency beating of sound waves

A violinist tries to tune the strings of his instrument.
a. Comparing the a-string ('la') to a tuning fork ($\nu_{dia} = 440\,\text{Hz}$), he hears a beat with the frequency $1\,\text{Hz}$. By increasing the tension on the rope, the beat frequency increases. What was the frequency of the 'a'-string before the tension increased?
b. After having adjusted the a-string the violinist wants to tune the d-string ('re'). He realizes that the second harmonic $3\nu_d$ produces with the first harmonic of the a-string ($2\nu_a$) a beat of $1\,\text{Hz}$. Decreasing the tension of the d-string the beat disappears. What was the initial frequency of the d-string and by what percentage does the violinist need to decrease the tension of the string?

5.4.6.10 Ex: Normal modes on a string

A stretched wire of mass m, length L, and tension T is triggered by two sources, one at each end. Both sources have the same frequency ν and amplitude A, but are out of phase by exactly 180° with respect to each other. (At each end there is an anti-node.) What is the lowest possible value of ω consistent with the stationary vibrations of the wire?

5.4.6.11 Ex: Normal modes on a string

a. Find the total vibration energy of a wire of length L fixed at both ends and oscillating in its n-th characteristic mode with amplitude A. The tension on the wire is T, and its total mass is M. (**Suggestion:** Consider the integrated kinetic energy at the instant when the wire is straight.)
b. Calculate the total vibration energy of the same wire vibrating in the following superposition of normal modes:

$$Y(x,t) = A_1 \sin \tfrac{\pi x}{L} \cos \omega_1 t + A_3 \sin \tfrac{3\pi x}{L} \cos(\omega_3 t - \tfrac{\pi}{4}) \; .$$

You should be able to verify that it is the sum of the energies of the two modes taken separately.

5.4.6.12 Ex: Normal modes on a string

A wire of length L is attached at both ends under a tension T. The wire is pulled sideways by a distance h from its center, such that the rope adopts a triangular shape, and the it is released.
a. What is the energy of the subsequent oscillations. **Suggestion**: Consider the work that needs to be done against the tension to give the wire its initial deformation, and suppose that the tension remains unchanged upon a slight increase of its length

caused by transverse the displacements.
b. How many times will the triangular shape reappear?

5.4.6.13 Ex: Waves on a rope

A string with linear mass density μ is attached at two points distant from each other by $L = 1\,\text{m}$. A mass $m = 1\,\text{kg}$ is now attached to one end of the rope that goes through a pulley, as shown in the figure. Excited by a vibrating pin with frequency $f = 1\,\text{kHz}$ the string performs transverse vibrations with wavelength $\lambda = 2L$.
a. Calculate the propagation velocity of the wave.
b. At what frequency should the pin excite the rope to observe the third oscillation mode (three anti-nodes)?
c. Now the mass is doubled. Calculate the new speed of sound.
d. How should the mass be chosen to obtain a fundamental mode frequency equal to the frequency of the third mode calculated in (b)?

Figure 5.27: Waves on a rope.

5.5 Matter waves

Quantum mechanics tells us that light sometimes behaves like particles and matter like waves. Letting us guide by this analogy we will, in the following, guess the fundamental equations of motion for the propagation of matter waves from a comparison of the respective dispersion relations of light and massive particles.

5.5.1 Dispersion relation and Schrödinger's equation

On one hand, the propagation light is (in the vacuum) is described by the dispersion relation $\omega = ck$ or,

$$\omega^2 - c^2k^2 = 0 \; . \tag{5.172}$$

Since light is a wave it can, in the most general form, be described by a wavepacket, $A(\mathbf{r}, t) = \int e^{\imath(\mathbf{k}\cdot\mathbf{r}-\omega t)}a(\mathbf{k})d^3k$. It is easy to verify that the *wave equation*,

$$\frac{\partial^2}{\partial t^2}A - c^2\nabla^2 A = 0 \; , \tag{5.173}$$

reproduces the dispersion relation.

On the other hand, slow massive particles possess the kinetic energy,

$$E = \frac{p^2}{2m} \ . \tag{5.174}$$

With de Broglie's hypothesis that even a massive particle has wavelength, we can try an *ansatz* [10] for a wave equation satisfying the dispersion relation (5.174). From Planck's formula, $E = \hbar\omega$, and the formula of *Louis de Broglie*, $\mathbf{p} = \hbar\mathbf{k}$, describing the particle by a wavepacket not being subject to external forces $\psi(\mathbf{r}, t) = \int e^{\imath(\mathbf{k}\cdot\mathbf{r}-\omega t)}\phi(\mathbf{k})d^3k$, it is easy to verify that the differential equation,

$$\imath\hbar\frac{\partial}{\partial t}\psi = \left(-\frac{\hbar^2}{2m}\nabla^2\right)\psi \ , \tag{5.175}$$

reproduces the dispersion relation. If the particle is subject to a potential, its total energy is $E = \mathbf{p}^2/2m + V(\mathbf{r}, t)$. This dispersion relation corresponds to the famous *Schrödinger equation*,

$$\imath\hbar\frac{\partial}{\partial t}\psi = \left(-\frac{\hbar^2}{2m}\Delta + V(r,t)\right)\psi \ . \tag{5.176}$$

Since we accept that light particles and lenses behave like a wave, to calculate their trajectories, we must determine the potential landscape $V(\mathbf{r})$ in which this particle moves before solving the Schrödinger equation. This is the role of *wave mechanics*, which is one of the formulations of quantum mechanics.

5.5.1.1 Scalar waves and vectorial waves

The electromagnetic field is a *vector field*, since $\vec{\mathcal{E}}(\mathbf{r}, t)$ and $\vec{\mathcal{B}}(\mathbf{r}, t)$ are vectors. Therefore, it has a polarization. In contrast, the field of matter $\psi(\mathbf{r}, t)$ is a *scalar field* and therefore does not have the degree of freedom of polarization, in analogy with sound. This has important consequences, for example, the fact that two collinear light fields with orthogonal polarizations do not interfere has no analogue with matter wave fields.

5.5.2 Matter waves

Broglie's formula assigns a wave to each body, The wavelength decreases as the velocity of the particle grows. The necessity to describe a massive particle as a matter wave depends on the relationship between its Broglie wavelength and other characteristic quantities of the system under consideration. If the wavelength is large, we expect typical interference phenomena for waves; if the wavelength is small, the particle will behave like a mass, which is perfectly localized in space and incapable of interfering.

Characteristic features of the system may be, for example, the presence of a narrow slit diffracting the Broglie wave of an atom or an electron passing through it. Another characteristic feature is the average distance between several atoms. In fact, when an atomic gas is so cold, that is, composed of atoms so slow, that the Broglie wavelength of the atoms is longer than the average distance, then the atoms interfere with each

[10] Kick, work hypothesis, guess.

other. In the case of bosonic atoms, the interference will be constructive, resulting in a matter wave of gigantic amplitude. This phenomenon is called Bose-Einstein condensation [11].

Before calculating the temperature required for this phenomenon to happen, we need to inform the reader, that the interatomic distance can not be compressed arbitrarily, because below distances of typically $\bar{d} = 1\ \mu$m, the gas tends to form molecules. For the Broglie waves of different atoms to interfere, the wavelength must be longer. The average velocity of the atoms in a gas of temperature T is given by,

$$\frac{m}{2}\bar{v}^2 = \frac{k_B}{2}T \ .$$

Therefore, the temperature of the gas must be,

$$T = \frac{m\bar{v}^2}{k_B} = \frac{\bar{p}^2}{k_B m} = \frac{\hbar^2\bar{k}^2}{k_B m} = \frac{4\pi^2\hbar^2}{k_B m\lambda_{dB}^2} < \frac{h^2}{k_B m d^2} \ .$$

For rubidium atoms of mass $m = 87u$ we calculate $T < 200\,\text{nm}$.

The development of powerful experimental techniques allowed in 1995 the cooling of rubidium gases down to such low temperatures and the experimental realization of Bose-Einstein condensates, that is, matter waves made up of 10^6 atoms. See Exc. 5.5.3.1.

5.5.3 Exercises

5.5.3.1 Ex: Interference in Bose-Einstein condensates

Calculate the periodicity of the interference pattern of two Bose-Einstein condensates supposed to have intrinsic temperatures $T = 0$ interpenetrating at a relative velocity $v = 1\,\text{mm/s}$.

5.6 Further reading

H.M. Nussenzveig, Edgar Blucher (2014), *Curso de Física Básica: Fluidos, Vibrações e Ondas, Calor - vol 2* [ISBN]

[11] See script on *Quantum mechanics* (2023).

Chapter 6

Gravitation

Gravity is one of the four fundamental forces known in modern physics. Just like the electromagnetic force it has a long range, but differently from it, it is always attractive. For these reasons it rules the macroscopic behavior of our universe. When we studied the motion and dynamics of bodies, until now we either considered contact interaction without worrying about their microscopic nature and assuming the validity of conservation laws, or we tacitly supposed the presence of force fields without worrying where they came from.

There is, however, a force acting between distant bodies without them having to touch each other. Newton introduced gravity as an *action at a distance*, meaning that all bodies instantaneously sense the presence and eventually the motion of all other bodies. Einstein showed later, that this concept cannot be right.

We will in this chapter give an introduction into Newton's theory and a very brief outlook an Einstein's theory of gravity.

6.1 Planetary orbits

6.1.1 Kopernicus' laws

Nicolaus Copernicus published in 1543 his book *De revolutionibus orbium coelestium* in which he states:

1. The planetary orbit is a circle with epicycles.

2. The Sun is approximately at the center of the orbit.

3. The speed of the planet in the main orbit is constant.

Despite being correct in saying that the planets revolved around the Sun, Copernicus was incorrect in defining their orbits. It was Kepler who correctly defined the orbit of planets as follows:

1. The planetary orbit is not a circle with epicycles, but an ellipse.

2. The Sun is not at the center but at a focal point of the elliptical orbit.

3. Neither the linear speed nor the angular speed of the planet in the orbit is constant, but the area speed is constant.

6.1.2 Kepler's laws

Kepler's laws of planetary motion, published by *Johannes Kepler* between 1609 and 1619, describe the orbits of planets around the Sun:

1. The orbit of a planet is an ellipse with the Sun at one of the two foci.
2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
3. The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit.

The elliptical orbits of planets were indicated by calculations of the orbit of Mars. From this, Kepler inferred that other bodies in the Solar System, including those farther away from the Sun, also have elliptical orbits. The second law helps to establish that when a planet is closer to the Sun, it travels faster. The third law expresses that the farther a planet is from the Sun, the slower its orbital speed, and vice versa. *Isaac Newton* showed in 1687 that relationships like Kepler's would apply in the Solar System as a consequence of his own laws of motion and law of universal gravitation. Do the Excs. 6.1.3.1 to 6.1.3.4.

The eccentricity of the orbit of the Earth makes the time from the March equinox to the September equinox, around 186 days, unequal to the time from the September equinox to the March equinox, around 179 days. A diameter would cut the orbit into equal parts, but the plane through the Sun parallel to the equator of the Earth cuts the orbit into two parts with areas in a 186 to 179 ratio, so the eccentricity of the orbit of the Earth is approximately,

$$e \approx \frac{\pi}{4}\frac{186-179}{186+179} \approx 0.015 \;, \tag{6.1}$$

which is close to the correct value (0.016710218). The accuracy of this calculation requires that the two dates chosen be along the elliptical orbit's minor axis and that the midpoints of each half be along the major axis. As the two dates chosen here are equinoxes, this will be correct when perihelion, the date the Earth is closest to the Sun, falls on a solstice. The current perihelion, near January 4, is fairly close to the solstice of December 21 or 22.

6.1.3 Exercises

6.1.3.1 Ex: Kepler orbits

The moon moves in a good approximation on a circular path with radius $R = 384000\,\text{km}$ around the Earth. Assume that the Earth's mass would suddenly decrease.
a. How much would the mass have to decrease so that the moon could escape the Earth?
b. How would the moon's orbit change if the mass decreased by a factor of 3, 2 or 1.5?

6.1.3.2 Ex: Kepler orbits of missiles

Consider an object of mass $m \ll M_{♁}$ which is launched at an initial velocity $\mathbf{v}_0$ (at an angle θ relative to the Earth's surface). We neglects any friction.
a. What possible trajectories can the object move on? How does the type of trajectory depend on the conservation parameters?
b. Calculate the maximum speed that the object may have to move on a closed trajectory. Does this speed depend on θ? Does the projectile always fall back to Earth when the path is closed?
c. Neglecting the Earth's rotation calculate the flight distance of the projectile above the Earth's surface for velocities below the above-mentioned limit velocity.
Help: Set the center of the Earth in the focal point of the Kepler orbit.

6.1.3.3 Ex: Halley's Comet

The comet Haley moves like a planet on an elliptical orbit around the sun. Its orbital period is 75 years and the closest distance to the sun is 0.5 AE. (One astronomical unit is the distance from the Earth to the sun, assuming that the orbit of the Earth around the sun is a circular orbit.)
a. Use this information to calculate the value for the major semi-axis a and the minor semi-axis b of the comet's orbit in astronomical units. Use this to determine the eccentricity ε of the orbit.
b. What is the maximum distance of the comet from the sun?
c. Calculate the minimum and maximum speed of the comet on its orbit.

6.1.3.4 Ex: Ellipse

An ellipse is the set of all points C satisfying the condition $\overline{AC} + \overline{BC} = 2a$, where the two focal points A and B are at a given distance $2e$.
a. Show that this definition is equivalent to the ellipse equation of the form,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ ,$$

where a and b are the large and small half-axes. How does e depend on a de b?
b. Show that the ellipse equation is given in plane polar coordinates by,

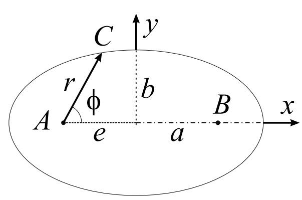

Figure 6.1: Ellipse.

$$r = \frac{P}{1 - \epsilon \cos \phi} \ .$$

Determine the eccentricity ϵ and the parameter P as a function of a and b.

6.2 Newton's law

Newton's law of *gravity* about the force between two massive bodies,

$$\mathbf{F} = -\nabla V(r) \ . \tag{6.2}$$

can be deduced from a conservative central potential,

$$\boxed{V(r) = \frac{\gamma_N Mm}{r}} \ . \tag{6.3}$$

If $M = M_♁$ is the mass of the Earth, a test mass m close to the surface ($r_♁ \approx 6378\,\text{km}$) will be accelerated by,

$$g = \frac{F}{m} = -\frac{\partial}{\partial r}\frac{\gamma_N M_♁}{r}\bigg|_{r=R_♁} = \frac{\gamma_N M_♁}{R_♁^2} = 9.81\,\text{m/s}^2 \ . \tag{6.4}$$

with Newton's constant,

$$\boxed{\gamma_N = 6.67 \cdot 10^{-11}\,\text{m}^3/\text{kg s}^2} \ . \tag{6.5}$$

6.2.1 Cosmic velocities

6.2.1.1 First cosmic velocity

The first *cosmic velocity* is defined as the velocity that a body must have in order to circle the center of the Earth on an orbit with the Earth's radius. We calculate this velocity from the condition that the centripetal force be equal to the centrifugal force,

$$\frac{mv_1^2}{r_♁} = \gamma_N \frac{mM_♁}{r_♁^2} \qquad \Rightarrow \qquad v_1 = \sqrt{\frac{\gamma_N M_♁}{r_♁}} \ , \tag{6.6}$$

yielding $v_1 \approx 7.91\,\text{km/s} = 2.84 \cdot 10^4\,\text{km/h}$. In Exc. 6.2.3.1 we estimate the mass of the milky way galaxy from the velocity of the sun and its distance from the galaxy's center. In Exc. 6.2.3.2 we compare the heights of stationary orbits around the Earth and the moon.

Example 24 (*Angular velocity of a satellite*): Here, we calculate the velocity of a satellite on a circular orbit at a height of 400 km above the Earth's surface,

$$v_1 = \sqrt{\frac{\gamma_N M_♁}{r_♁ + h}} \ ,$$

yielding $v_1 \approx 7.66\,\text{km/s} = 2.76 \cdot 10^4\,\text{km/h}$.

6.2.1.2 Escape velocity

The *escape velocity* or second cosmic velocity is the velocity that a body must have to be able to leave the Earth's gravity field completely. We calculate the second cosmic speed for the Earth from,

$$E_{kin} = \frac{m}{2}v_2^2 = \text{final} - \text{initial energy in the limit final energy} \ \to 0 \ . \tag{6.7}$$

Hence,

$$E_{kin} = 0 - \left(-\gamma_N \frac{mM_♁}{r_♁}\right) \qquad \Rightarrow \qquad v_2 = \sqrt{\frac{2\gamma_N M_♁}{r_♁}} = v_1\sqrt{2} \ , \tag{6.8}$$

yielding $v_2 \approx 11.2\,\text{km/s} = 4.03 \cdot 10^4\,\text{km/h}$. Apparently, the cosmic velocities v_1 and v_2 are related. In Exc. 6.2.3.3 and 6.2.3.4 we calculate cosmic velocities for, respectively, Earth and the comet Tschurjumow-Gerasimenko.

> **Example 25 (*Escape velocity for a satellite*):** The escape velocity for a satellite that is in a 400 km high orbit above the Earth's surface is $v_2 \approx 10.83\,\text{km/s} = 3.90 \cdot 10^4\,\text{km/h}$.

6.2.2 Deriving Kepler's laws from Newton's laws

6.2.2.1 Kepler's first law

The orbits are ellipses, with focal points F1 and F2 for the first planet and F1 and F3 for the second planet. The Sun is placed at focal point F1. The two shaded sectors A1 and A2 have the same surface area and the time for planet 1 to cover segment A1 is equal to the time to cover segment A2. The total orbit times for planet 1 and planet 2 have a ratio $(a_1/a_2)^{3/2}$.

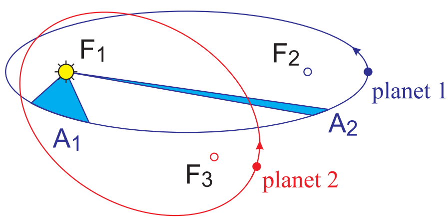

Figure 6.2: Illustration of Kepler's three laws with two planetary orbits.

6.2.2.2 Kepler's second law

The area swept by the planet's trajectory in infinitesimal time steps is,

$$A(t, t + dt) = \tfrac{1}{2}|\mathbf{r}(t) \times \dot{\mathbf{r}}(t)|dt = \tfrac{L}{2m}dt \ .$$

Since central potentials preserve angular momentum,

$$\dot{\mathbf{L}} = \frac{d}{dt}m\mathbf{r} \times \mathbf{p} = m(\dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}}) = m\mathbf{r} \times \ddot{\mathbf{r}} = -\mathbf{r} \times \nabla V(r) = -\mathbf{r} \times \frac{\partial V(r)}{\partial r}\hat{\mathbf{e}}_r = \mathbf{0} \ ,$$

for a given time difference $dt = t_1 - t_0$ the swept area is the same. Angular momentum is a constant of motion, $\dot{\mathbf{L}} = 0$, for central potentials.

6.2.2.3 Kepler's third law

6.2.3 Exercises

6.2.3.1 Ex: Mass of the Milky Way

Estimate the total mass of our galaxy (the milky way) using the parameters of the orbits of the sun (and the solar system) around the center of the galaxy. Assume that the major part of the mass of our galaxy is in the form of a uniform sphere (bulge). The speed of the sun on its way around the center of the galaxy is approximately $v = 250\,\text{km/h}$, the distance of the sun from the center of the galaxy is approximately $r = 28000\,\text{ly}$ (light years). To how many stars like our sun does this correspond to?

6.2.3.2 Ex: Gravitation on Earth and Moon

How high are the orbits of 'geo-stationary' and 'lunar-stationary' satellites?

6.2.3.3 Ex: Cosmic velocities

a. How long is the orbital period T of a $1\,\text{t}$ satellite on a circular orbit at a height of $20\,\text{km}$ around the Earth? How long is the orbital period T of the Earth around the sun (the mass of the sun is 3.334×10^5 times larger than that of the Earth)? At what distance from Earth is the orbit of a satellite geostationary?
b. Calculate the escape velocity from Earth (or cosmic speed) for a person weighing $75\,\text{kg}$.

6.2.3.4 Ex: Tschurjumow-Gerasimenko

The satellite Rosetta of the ESA ($m_{sat} = 3000\,\text{kg}$) was placed on an orbit of the comet Tschurjumow-Gerasimenko (mass $m_{TG} = 3.14 \cdot 10^{12}\,\text{kg}$, diameter $d_{TG} = 4\,\text{km}$).
a. For the satellite to orbit the comet once a terrestrial day, what is the required height of the orbit?
b. What is the escape velocity from the comet's surface?

6.3 Gravitational potential

For an arbitrary mass distribution $\rho(\mathbf{r})$ the *gravitational potential* acting on a test mass m can be calculated from,

$$\boxed{V(\mathbf{r}) = -\gamma_N m \int_{\mathbb{R}^3} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'} \ . \tag{6.9}$$

For a point-mass with mass M located at the origin, $\mathbf{r}' = 0$, we parametrize $\rho(\mathbf{r}') = M\delta^3(\mathbf{r}')$, and recover Newton's law,

$$V(\mathbf{r}) = -\gamma_N \frac{Mm}{r} \ . \tag{6.10}$$

The gravitational potential being conservative, trajectories of test masses can simply be derived by solving the equation of motion,

$$\boxed{m\ddot{\mathbf{r}} = -\nabla V(\mathbf{r}) = \gamma_N m \int_{\mathbb{R}^3} \rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3r'} \ . \tag{6.11}$$

If in practice analytic solution are beyond reach, numerical procedure are always possible.

Example 26 (*Gravitational potential in- and outside a homogeneous sphere*): In this example we will calculate the gravitational force that a particle of mass m is subjected to when placed inside a homogeneous sphere of radius R at a distance r from its center.
The potential exerted by a mass distribution with the density $\rho(\mathbf{r}')$ on a particle of mass m located at the position $\mathbf{r}$ is,

$$V(\mathbf{r}) = -\int \rho(\mathbf{r}')\frac{\gamma_N m}{|\mathbf{r}-\mathbf{r}'|}d^3r' = -\int_{sphere} \rho_0 \frac{\gamma_N m}{|\mathbf{r}-\mathbf{r}'|} r'^2 \sin\theta' dr' d\theta' d\phi' \ . \tag{6.12}$$

Substituting,

$$\begin{aligned} \xi &\equiv |\mathbf{r}-\mathbf{r}'| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta'} \\ \frac{d\xi}{d\theta'} &= \frac{rr'\sin\theta'}{\xi} \ , \end{aligned} \tag{6.13}$$

we obtain,

$$V(\mathbf{r}) = -\int_{sphere} \rho_0 \frac{\gamma_N m r'}{r} d\xi dr' d\phi' = \frac{2\pi\rho_0\gamma_N m}{r}\int_0^R \int_{\xi_{\min}}^{\xi_{\max}} r' d\xi dr' \ . \tag{6.14}$$

The integration limits follow from the values adopted by ξ for $\theta = 0$ resp. $\theta = \pi$. For $r \leq R$ we have that r' is always greater than r. Hence, $\xi = r' - r, .., r' + r$. For $R \leq r$ we have that r' is always smaller than r. Hence, $\xi = r - r', .., r' + r$.

$$V(\mathbf{r}) = -\frac{2\pi\rho_0\gamma_N m}{r}\begin{cases} \int_r^R 2rr'dr' + \int_0^r 2r'^2 dr' \\ \int_0^R 2r'^2 dr' \end{cases} \quad \text{for} \quad \begin{cases} r \leq R \\ R \leq r \end{cases} \ . \tag{6.15}$$

With the sphere's mass,

$$M = \frac{4\pi\rho_0 R^3}{3} \ , \tag{6.16}$$

the potential becomes,

$$\begin{aligned} V(\mathbf{r}) &= -2\pi\rho_0\gamma_N m\left(R^2 - \frac{1}{3}r^2\right)\theta(R-r) - 2\pi\rho_0\gamma_N m\frac{2R^3}{3r}\theta(r-R) \\ &= -\gamma_N M m\left(\frac{3}{2R} - \frac{r^2}{2R^3}\right)\theta(R-r) - \gamma_N M m\frac{1}{r}\theta(r-R) \ . \end{aligned} \tag{6.17}$$

The force can be calculated using the gradient in spherical coordinates,

$$\begin{aligned} \mathbf{F} &= -\nabla V(\mathbf{r}) = -\hat{\mathbf{e}}_r\frac{\partial}{\partial r}V(\mathbf{r}) \\ &= -\hat{\mathbf{e}}_r\gamma_N M m\frac{r}{R^3}\theta(R-r) - \hat{\mathbf{e}}_r\gamma_N M m\frac{1}{r^2}\theta(r-R) \ . \end{aligned} \tag{6.18}$$

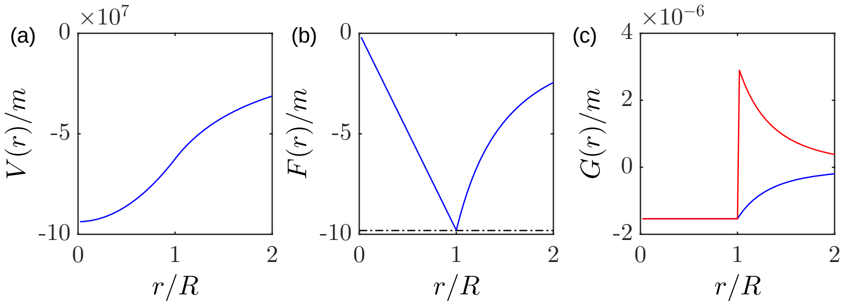

Figure 6.3: (code) Gravity in- and outside of Earth. (a) Gravitational potential, (b) gravitational force, and (c) radial (blue) and transverse (red) gravity gradient.

The gravitational acceleration is often specified in units of *Gal* after *Galilei*, where $1\,\text{Gal} \equiv 1\,\text{cm/s}^2$.

The above example shows that

1. outside a spherical mass distribution the gravitational potential can simply be replaced by that of a point mass sitting at the center of the mass distribution;

2. the superposition principle,

$$\boxed{V_{\rho_1+\rho_2}(\mathbf{r}) = V_{\rho_1}(\mathbf{r}) + V_{\rho_2}(\mathbf{r})} , \tag{6.19}$$

allows us to described the impact of mass cavities via simple subtraction

In classical mechanics we often describe gravity as a homogenous force field, which can be derived from a potential scaling linearly with the height above normal ground,

$$V(h) = mgh . \tag{6.20}$$

Obviously, this is approximation obtained by linearizing the gravitational potential on the Earth's surface. From Newton's law,

$$V(\mathbf{r}) = -\frac{\gamma_N Mm}{r} , \tag{6.21}$$

using the Taylor expansion:

$$V(\mathbf{r}+\mathbf{h}) = e^{\mathbf{h}\cdot\nabla_\mathbf{r}}V(\mathbf{r}) = \sum_{\nu=0}^{\infty}\frac{(\mathbf{h}\cdot\nabla_\mathbf{r})^\nu}{\nu!}V(\mathbf{r}) = V(\mathbf{r})+(\mathbf{h}\cdot\nabla_\mathbf{r})V(\mathbf{r})+\frac{1}{2}(\mathbf{h}\cdot\nabla_\mathbf{r})(\mathbf{h}\cdot\nabla_\mathbf{r})V(\mathbf{r}) , \tag{6.22}$$

we get,

$$V(\mathbf{r}+\mathbf{h}) \simeq V(\mathbf{r}) + h\frac{\gamma_N Mm}{r^2} = V(\mathbf{r}) + hgm . \tag{6.23}$$

In Exc. 6.3.5.1 we derive an expression generalizing Eq. (6.17) to arbitrary isotropic gravitational potentials. In Excs. 6.3.5.2, 6.3.5.3, 6.3.5.4, and 6.3.5.5 we calculate the potentials for other isotropic mass distributions. In Exc. 6.3.5.6 we use the superposition principle to calculate the potential generated by a spherical cavity inside a homogeneous sphere. In Excs. 6.3.5.7, 6.3.5.8, and 6.3.5.9 we calculate potentials generated by non-spherical density distributions. In Excs. 6.3.5.10, 6.3.5.11, and 6.3.5.12 we apply the results derived for the Earth's inner gravitational potential to derive possible trajectories through boreholes traversing the Earth.

6.3.1 Rotation and divergence of gravitational force fields

The rotation and divergence of gravitational force fields are,

$$\begin{aligned} \nabla^2 V(\mathbf{r}) = \nabla\cdot\mathbf{F}(\mathbf{r}) &= -4\pi\gamma_N m\rho(\mathbf{r}) \\ \nabla\times\mathbf{F}(\mathbf{r}) &= 0 \ . \end{aligned} \tag{6.24}$$

The integral formulation of Eq. (6.24) reads,

$$\oint_{\partial V}\mathbf{F}\cdot d\mathbf{S} = -4\pi\gamma_N Mm \ , \tag{6.25}$$

with $M = \oint_{\partial V}\rho(\mathbf{r})d^3r$.

The interpretation of these expressions are:

- The Poisson equation relates the divergence of the force field directly to the density distribution.
- The divergence is nothing else than the diagonal of gravity gradient defined in Sec. 6.3.2.
- For being conservative, gravitational potentials are rotation-free.
- The integral over a closed surface is proportional to the enclosed mass.

The Lagrangian density for Newtonian gravity is,

$$\mathcal{L}(\mathbf{r},t) = -\rho(\mathbf{r},t) - \tfrac{1}{8\pi\gamma_N}[\nabla V(\mathbf{r},t)]^2 \ . \tag{6.26}$$

Applying the Hamitonian principle to this Lagrangian one recovers the Poisson equation for gravity.

6.3.2 Gravity gradients

The *gravity gradient* is a tensor defined as the second derivative of the potential,

$$\boxed{G_{kl}(\mathbf{r}) = G_{lk}(\mathbf{r}) = \frac{\partial g_l(\mathbf{r})}{\partial x_k} = \frac{1}{m}\frac{\partial F_l(\mathbf{r})}{\partial x_k} = -\frac{1}{m}\frac{\partial}{\partial x_k}\frac{\partial V(\mathbf{r})}{\partial x_l}} \ . \tag{6.27}$$

Inserting the potential (6.9) we obtain,

$$\begin{aligned} G_{kl}(\mathbf{r}) &= -\frac{1}{m}\frac{\partial}{\partial x_k}\frac{\partial V(\mathbf{r})}{\partial x_l} = \gamma_N\int_{R^3}\rho(\mathbf{r}')\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l}\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3r' \\ &= \gamma_N\int_{R^3}\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^5}\begin{pmatrix} 3(x-x')^2-(\mathbf{r}-\mathbf{r}')^2 & 3(x-x')(y-y') & 3(x-x')(z-z') \\ 3(x-x')(y-y') & 3(y-y')^2-(\mathbf{r}-\mathbf{r}')^2 & 3(y-y')(z-z') \\ 3(x-x')(z-z') & 3(y-y')(z-z') & 3(z-z')^2-(\mathbf{r}-\mathbf{r}')^2 \end{pmatrix} d^3r' \\ &= \gamma_N\int_{R^3}\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}K_{kl}(\mathbf{r}-\mathbf{r}')d^3r' \ , \end{aligned} \tag{6.28}$$

defining the kernel,

$$K_{kl}(\mathbf{r}-\mathbf{r}') \equiv \frac{3(x_k-x_k')(x_l-x_l')-\delta_{kl}(\mathbf{r}-\mathbf{r}')^2}{|\mathbf{r}-\mathbf{r}'|^2} \ . \tag{6.29}$$

For example, for the gravitational potential generated by a point mass,

$$V(r) = \gamma_N \frac{Mm}{r} = \gamma_N \frac{Mm}{\sqrt{x^2+y^2+z^2}} \ , \tag{6.30}$$

we find,

$$\begin{aligned} G_{kl}(\mathbf{r}) &= -\frac{1}{m}\frac{\partial}{\partial x_k}\frac{\partial V(\mathbf{r})}{\partial x_l} = \frac{\gamma_N M}{r^5}\begin{pmatrix} 3x^2-r^2 & 3xy & 3xz \\ 3xy & 3y^2-r^2 & 3yz \\ 3xz & 3yz & 3z^2-r^2 \end{pmatrix} \\ &= \frac{\gamma_N M}{r^5}(3x_kx_l - r^2\delta_{kl}) \ . \end{aligned} \tag{6.31}$$

Gravity gradients are often given in units of *Eotvos* after *Eötvös*, where 1 Eotvos = $10^{-9}\,\text{s}^{-2}$. In Exc. 6.3.5.13 we calculate the gravity gradient tensor of Earth (modeled as an idealized sphere) at the north-pole.

Example 27 (*Gravitational curvature in- and outside a homogeneous sphere*): The gravitational potential an force in- and outside a homogeneous sphere have been calculated in the example 26. Using the result we derive the gravity gradient,

$$\begin{aligned} G_{kl}(\mathbf{r}) &= -\frac{1}{m}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l}V(\mathbf{r}) \\ &= -\frac{\gamma_N M}{R^3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\theta(R-r) + \frac{\gamma_N M}{r^5}\begin{pmatrix} 3x^2-r^2 & 3xy & 3xz \\ 3xy & 3y^2-r^2 & 3yz \\ 3xz & 3yz & 3z^2-r^2 \end{pmatrix}\theta(r-R) \\ &= -\frac{\gamma_N M}{R^3}\delta_{kl}\theta(R-r) - \frac{\gamma_N M}{r^3}\left(\delta_{kl} - \frac{3x_kx_l}{r^2}\right)\theta(r-R) \ . \end{aligned} \tag{6.32}$$

The example 26 revealed that neither the potential nor the force are discontinuous at the sphere's surface. In contrast, the radial component of the curvature $G_{k=l}(r=R)$ is discontinuous at the north pole, while the transverse components $G_{k\neq l}(r=R)$ stay continuous, which is obviously due to the isotropic symmetry of the potential. To see this better, let us move along the symmetry axis setting $\mathbf{r} = r\hat{\mathbf{e}}_z$,

$$G_{kl}(r\hat{\mathbf{e}}_z) = -\frac{\gamma_N M}{R^3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\theta(R-r) - \frac{\gamma_N M}{r^3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\theta(r-R) \ . \tag{6.33}$$

Applying this results to Earth, be find inside Earth a constant gravity gradient of $-\gamma_N M_\oplus/R_\oplus^3 = 1.54 \cdot 10^{-6}\,\text{s}^{-2}$.

6.3.2.1 Gravimetry and gravity gradiometry

Gravity-gradiometers measure spatial variations of the gravitational acceleration. Being obtained as second derivatives of the gravitational potential, they are more sensitive to local mass variations, as nearly homogeneous large scale contributions to the acceleration are removed. For this reason, gravity-gradiometers need to be less accurate, provided they are sensitive enough. In Exc. 6.3.5.14 we estimate the sensitivity of modern gravimeters.

Example 28 (*Gravitation in- and outside a massive shell*): The calculations of examples 26 and 27 can be generalized for a homogenous massive shell with density ρ_1, inner radius R_i, and outer radius R_0. In Exc. 6.3.5.2 we show that the gravitational potential is,

$$V(\mathbf{r}) = -2\pi\rho_1\gamma_N m\left[(R_o^2 - R_i^2)\theta(R_i - r) + \left(R_o^2 - \frac{r^2}{3} - \frac{2R_o^3}{3r}\right)\theta(r - R_i)\theta(R_o - r) + \frac{2(R_o^3 - R_i^3)}{3r}\theta(r - R_o)\right] , \tag{6.34}$$

the gravitational force,

$$\begin{aligned}\mathbf{F}(\mathbf{r} &= -\nabla V(\mathbf{r}) = -\hat{\mathbf{e}}_r\frac{\partial}{\partial r}V(\mathbf{r}) \\ &= \hat{\mathbf{e}}_r\frac{4\pi\rho_1\gamma_N m}{3}\left[\left(-r + \frac{R_o^3}{r^2}\right)\theta(r - R_i)\theta(R_o - r) - \frac{R_o^3 - R_i^3}{r^2}\theta(r - R_o)\right] ,\end{aligned} \tag{6.35}$$

and the gravity gradient,

$$\begin{aligned}G_{kl}(\mathbf{r}) &= -\frac{1}{m}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l}V(\mathbf{r}) \\ &= -\frac{4\pi\rho_1\gamma_N}{3}\left[\left(\delta_{kl} - \frac{R_i^3}{r^3}\left(\delta_{kl} - \frac{3x_k x_l}{r^2}\right)\right)\theta(r - R_i)\theta(R_o - r) \right. \\ &\quad \left. + \frac{R_o^3 - R_i^3}{r^3}\left(\delta_{kl} - \frac{3x_k x_l}{r^2}\right)\theta(r - R_o)\right] .\end{aligned} \tag{6.36}$$

For $R_i \to 0$ we recover the results of example 27. Particularly along the symmetry axis,

$$G_{zz}(\mathbf{r}) = -\frac{4\pi\rho_1\gamma_N}{3}\left[\left(1 + \frac{2R_i^3}{r^3}\right)\theta(r - R_i)\theta(R_o - r) - 2\frac{R_o^3 - R_i^3}{r^3}\theta(r - R_o)\right] . \tag{6.37}$$

6.3.3 Constants of motion

Trajectories can also be derive exploiting constants of motion. In Excs. 6.3.5.18 to 6.3.5.23 we calculate trajectories of bodies under the influence of gravity.

6.3.4 The virial law

The virial law states,

$$\boxed{\overline{T} = -\tfrac{1}{2}\overline{\sum_i \mathbf{F}_i\mathbf{r}_i}} . \tag{6.38}$$

For potentials of the form $V(r) = \alpha r^k$ we have,

$$\mathbf{F} = -\nabla V = -k\alpha r^{k-1}\hat{\mathbf{e}}_r \ . \tag{6.39}$$

Thus $\overline{T}$ and $\overline{V}$ related via,

$$\overline{T} = -\tfrac{1}{2}\overline{\sum_i \mathbf{F}_i \mathbf{r}_i} = \tfrac{k}{2}\overline{\sum_i \alpha r_i^{k-1}\hat{\mathbf{e}}_{r_i}\cdot\mathbf{r}_i} = \tfrac{k}{2}\overline{\sum_i \alpha r_i^k} = \tfrac{k}{2}\overline{V} \ . \tag{6.40}$$

In Exc. 6.3.5.24 we apply the virial law to a spring pendulum.

Example 29 (*The virial law for the harmonic potential and for $1/r$-potentials*): The special case $k = 2$ yields,

$$V(r) = \alpha r^2 \Rightarrow \overline{T} = \overline{V} \ , \tag{6.41}$$

and corresponds to a harmonic oscillator with $\alpha = \frac{1}{2}m\omega_0^2$.
The special case $k = -1$ yields,

$$V(r) = \frac{\alpha}{r} \Rightarrow \overline{T} = -\frac{1}{2}\overline{V} \ , \tag{6.42}$$

and corresponds to a Coulomb potential with $\alpha = \frac{q_1 q_2}{4\pi\epsilon_0}$, respectively a gravitational potential with $\alpha = -\gamma_N Mm$.
In the case of the gravitational potential, for positive total energy, we get,

$$E = \overline{T} + \overline{V} > 0 \qquad \overline{T} = -\tfrac{1}{2}\overline{V} \Rightarrow E = \tfrac{1}{2}\overline{V} > 0 \Rightarrow Mm < 0 \ . \tag{6.43}$$

Thus,

$$\overline{T} = \frac{1}{2}m\overline{v^2} = -\frac{1}{2}\overline{V} < 0 \Rightarrow m < 0 \tag{6.44}$$

This leads to the demand for negative masses, which is not sensible. The virial theorem can only apply to bound systems with $E < 0$.

6.3.5 Exercises

6.3.5.1 Ex: Arbitrary isotropic mass density distributions

a. Generalize the calculation of gravitational potentials and forces exhibited in example 25 to arbitrary, but isotropic mass density distributions $\rho(\mathbf{r}') = \rho(r')$.
b. Study the case of a sharp edge, $\rho(r') \equiv \rho(r')\theta(R - r')$.
c. Study the case of a homogeneous distribution, $\rho(r') \equiv \rho_0\theta(R - r')$, for a sphere with total mass M.
d. Study the case of a parabolic distribution, $\rho(r') \equiv \rho_0\left(1 - \frac{r'^2}{R^2}\right)\theta(R - r')$, for a sphere with total mass M.

6.3.5.2 Ex: Gravitational potential of a spherical shell

Consider a spherical shell with an inner radius a and an outer radius b.
a. Calculate the gravitational potential inside the sphere, inside the shell material and outside the sphere. (**Help:** Substitute the distance between the test particle m and a point of the mass distribution and make a case distinction for the integration limits for this distance variable.)
b. Calculate the force on a test particle.
c. Specify now for a massive sphere.
d. Specify for a very thin spherical shell.

6.3.5.3 Ex: Two concentric shells

Let us consider two concentric spherical shells of uniform density with masses M_1 and M_2. Calculate the force on a particle of mass m placed (a) inside the inner shell, (b) outside the inner but inside the outer shell, and (c) outside the outer sphere.

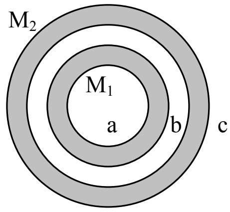

Figure 6.4:

6.3.5.4 Ex: Gravity influenced by a thin surface layer

Model the Earth as a homogeneous sphere of mass density ρ_0 isotropically covered by a $\Delta R = 1$m thick homogeneous layer with different density ρ_1. How does the gravitational potential depend on the ratio ρ_1/ρ_0?

6.3.5.5 Ex: Gravitational force inside a shell

Show through geometric arguments that a particle of mass m placed inside a spherical shell of uniform mass density is subject to zero force, regardless of the position of the particle. What would happen if the surface mass density was not constant?

6.3.5.6 Ex: Gravitational potential of a massive sphere with spherical cavity

A spherical cavity is machined into in a lead sphere of radius R such that its surface touches the outer surface of the massive sphere and passes through the its center. The primitive mass of the lead sphere is M. What will be the force that the sphere with the cavity will exert on a mass m at a distance z from the center of the outer sphere, when the mass and the centers of the sphere and the cavity are aligned?

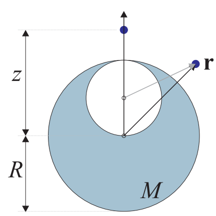

Figure 6.5: Scheme of the problem.

6.3.5.7 Ex: Gravitational potential of a disk

Calculate the potential of a homogeneous thin disc with the surface density $\sigma = M/\pi R^2 = \rho dz$ along the axis of symmetry and the gravitational force it exerts on a mass m.
Help: When integrating over the thickness a of the disk, use the relation: $\int_0^d zf(z')dz' = f(0)dz$.

6.3.5.8 Ex: Gravitational force of a ring

Calculate the gravitational force of a ring of linear mass density $\lambda = M/2\pi R = \rho dRdz$ on the symmetry axis.
Help: When integrating on the thickness of the ring, use the relations: $\int_0^{dz} f(z')dz' = f(0)dz$ and $\int_R^{R+dR} f(r')dr' = f(R)dR$.

6.3.5.9 Ex: Gravitational oscillation through a ring

Consider a heavy ring of mass M and radius R and a particle of mass m placed in its center. What is the frequency for small amplitude oscillations in the direction perpendicular to the plane of the ring?

6.3.5.10 Ex: Intraplanetary oscillation

A body of mass m is placed at a distance r_0 from the center of a planet of mass M and radius R.
a. Calculate the potential energy for $0 \leq r \leq \infty$. Suppose that the mass density of the planet is uniform and that the mass m can move within it through a tunnel. Consider $V(\infty) = 0$. Calculate the velocity as a function of r for $r < R$ knowing that $V(r_0) = 0$.

6.3.5.11 Ex: Shortcut avoiding the Earth's center

Show that in a tunnel dug through the Earth (not necessarily along a diameter) the movement of an object will be harmonic.

6.3.5.12 Ex: Shortcut through the Earth

a. Two innovative companies make suggestions on how to get mail to New Zealand as quickly as possible. One company suggests drilling a hole through the Earth, placing the mail in a fireproof box and allowing it to swing through the hole (smoothly) through the center of the Earth so that it can be easily received by the recipient in New Zealand. The other company wants to shoot the mail in a very low orbit of only 1 m above the surface of the Earth at the first cosmic speed (smoothly) to New Zealand, where it should then be caught by a correspondingly soft pillow. Which of these two suggestions (if they were feasible) would get the mail faster to destination?
b. Assume that the well was planned incorrectly and that the hole missed the center of the earth by 100 km. What does the equation of motion look like?
Help: The mass distribution of the earth can be assumed to be homogeneous. Earth rotation and friction effects are neglected.

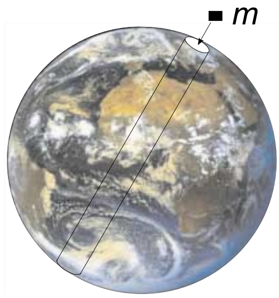

Figure 6.6:

6.3.5.13 Ex: Gravity gradient caused by underground cavities

In this exercise we discuss whether gravity gradiometry can identify the presence of underground cavities. We proceed in steps:
a. Assuming a homogeneous density distribution for Earth, calculate the gravity gradient tensor at the north-pole.
b. How does the tensor change in the presence of a point-like mass $M_1 = 10\,\text{tons}$ located at a distance $d = 1\,\text{m}$ in southern direction.
c. Describe the underground cavity by a spherical void centered at 1 m below the north-pole's surface at having a radius such that the missing mass corresponds to 10 tons of Earth material.

6.3.5.14 Ex: Gravity gradient upon horizontal density modulation

A gravimeter and a gravity gradiometer are hovering at a distance z over a flat surface infinitely extended in x and y. The surface has a thickness b and a horizontal density modulation in x-direction: $\rho_1(\mathbf{r}') = \rho_1(x')\theta(b+z')\theta(-z')$. Calculate the gravitational potential, force, and gradients. Consider the particular case of a localized density modulation: $\rho_1(x') = \rho_1\theta(x')\theta(a-x')$.

6.3.5.15 Ex: Gravity gradients

a. Modern commercial gravity gradiometers can measure acceleration gradients on the order of $|\nabla a| \approx 10^{-5}\,\text{s}^{-2}$. Compare with the gravity gradient on the Earth's surface. What is the smallest height difference detectable by a state of the art gradiometer?
b. Calculate the gravity gradient caused by a massive sphere of mass $m_{sphere} = 10\,\text{t}$ at $d = 1\,\text{m}$ distance?
c. The French company μQuans offers atomic quantum gravimeters with guaranteed sensitivities of $50\,\mu\text{Gal}/\sqrt{\text{Hz}}$ at a cycling frequency of 2 Hz. Assuming the Earth as a homogeneous sphere, for how long must the signal be integrated to be able to measure a 1 cm height variation over the Earth's surface.
d. For how long must the signal be integrated to be able to measure a gravity variation caused by a 10 t mass at 1 m distance.

6.3.5.16 Ex: Acceleration of a mass subject to a circular motion in an inhomogeneous force field

Commercial *Gravity Gradient Instruments* (GGI) are based on accelerometers mounted on the border of a disk of radius R rotating at a frequency ω. Let us suppose that the disk's rotation axis is the z-axis and that is located inside an inhomogeneous force field (e.g. gravity) characterized by its gradient tensor (assumed to be constant over time and over the length scale of R).
a. Calculate the time-dependent acceleration recorded by the accelerometer in radial direction.
b. The voltage signal delivered by the accelerometer is now added to one delivered by a second accelerometer sitting on the opposite side of the disk.
c. Finally, the signals are demodulated at 2ω and time-averaged over a period $2\pi/\omega$.

6.3.5.17 Ex: Angular momentum in spherical coordinates

a. Calculate the acceleration, the angular momentum, and its derivative in spherical coordinates using the result of Exc. ??.
b. Set $\theta = \frac{\pi}{2}$ in all expressions.
c. Derive the equation of motion for a central potential.

6.3.5.18 Ex: Scattering at a central force, angular momentum

Consider the scattering of a particle of mass M at an attractive central force field $\mathbf{F}(\mathbf{r}) = -\frac{\alpha}{r^2}\hat{\mathbf{e}}_r$ with $\alpha > 0$. Far from the force center the velocity of the particle is given by v_∞. The asymptotic distance perpendicular to the velocity for very large distances from the force center is called the impact parameter b.
a. Determine the relationship between the impact parameter b and the angular momentum L of the particle.
b. The path of the particle has the shape of a conic section, which in plane polar coordinates can be parametrized by $r = P/(1 - \epsilon\cos\phi)$. Find ϵ and P as a function of b, v_∞, M and α. c. Find an expression for $\sin(\theta/2)$. Here, θ is the scattering

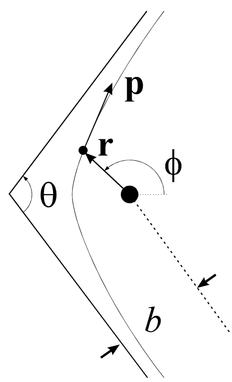

Figure 6.7:

angle between the asymptotic orbits of the particle, i.e. the paths of the incoming and

outgoing particles for large distances from the force center.
d. How does θ for constant v_∞ depend on the impact parameter b? Discuss the special cases $b = 0$ and $b \to \infty$.

6.3.5.19 Ex: Gravitational force, trajectory

The trajectory of the Kepler problem can be derived from the integral expression:

$$\varphi(r) = \varphi_0 + \int_{r_0}^{r} \frac{l\,dr}{r^2\sqrt{2m(E+\frac{\alpha}{r}) - \frac{l^2}{r^2}}} \ .$$

Here E is the total energy, $\alpha = \gamma_N mM$ and $l = mr^2\dot{\varphi}$. We also introduce the quantities:

$$p = \frac{l^2}{m\alpha} \qquad \text{and} \qquad \varepsilon = \sqrt{1 + \frac{2El^2}{m\alpha^2}} \ .$$

a. Convince yourself that, with the substitution $\xi = (p/r - 1)/\varepsilon$, the integral expression can be written in the form:

$$\varphi(r) = \varphi_0 - \int_{\frac{1}{\varepsilon}(\frac{p}{r_0}-1)}^{\frac{1}{\varepsilon}(\frac{p}{r}-1)} \frac{d\xi}{\sqrt{1-\xi^2}} \ .$$

b. Show with the help of energy conservation that the minimum distance from the force center is determined by $r_{\min} = \frac{p}{1+\varepsilon}$ for all values of E.
c. Show that the trajectories for $\varphi_0 = \pi$ and $r = r_{\min}$ are $r(\varphi) = \frac{p}{1-\varepsilon\cos\varphi}$, where $\int dx/\sqrt{1-x^2} = \arcsin x$.
d. Confirm that for elliptical trajectories ($0 \le \varepsilon < 1$ and $p = b^2/a^2$ where $a(b)$, the major semi-axis follows Kepler's 3rd law. Use the area theorem.

6.3.5.20 Ex: Central force, trajectory

Consider two masses m_1 and m_2 located at $\mathbf{r}_1$ and $\mathbf{r}_2$. There is an attractive force between them of the amount $F(\mathbf{r}_1, \mathbf{r}_2) = 2\lambda/|\mathbf{r}_1 - \mathbf{r}_2|^3$ ($\lambda > 0$).
a. Specify the angular momentum of the relative motion $\mathbf{l}$ and the energy conservation as a function of $\mathbf{r}$, $\mathbf{p}$ and the reduced mass μ, whereby we may designate by $E > 0$ the total energy of the system.
b. At the time $t = 0$ we let the relative distance of both particles be $r_{\min}$, the relative velocity in the direction of r be zero and $\varphi(r_{\min}) = 0$. Determine the relationship between $r_{\min}$, E, l, λ and μ. Is it possible to eliminate l and λ from the energy conservation law? Calculate the function $r(t)$.
c. Express $\frac{d}{d\varphi}r(\varphi) = \dot{r}/\dot{\varphi}$ as a function of E, l, r and $r_{\min}$ and calculate the trajectory $r(\varphi)$.

6.3.5.21 Ex: Ballistic movement

Consider the movement of an intercontinental missile launched at an inclination of θ_0, as shown in the figure, with speed v_0, in the indicated position. Calculate the body's trajectory.

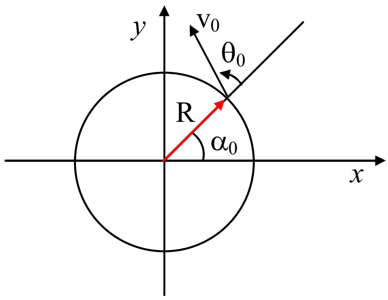

Figure 6.8:

6.3.5.22 Ex: Rotation of three bodies

Three identical bodies of mass M are located at the vertices of an equilateral triangle with border length L. How fast should they move, if they all rotate under the influence of mutual gravity, on a circular orbit that circumscribes the triangle always kept equilateral?

6.3.5.23 Ex: Rotation of two bodies

Consider two masses m and $2m$ with gravitational attraction. At what angular velocity should they rotate so that the distance d between them is constant?

6.3.5.24 Ex: The virial law

Consider a mathematical spring pendulum with $D = 100\,\text{N/m}$ and an attached mass of $m = 100\,\text{g}$. The average kinetic energy of the pendulum be $\overline{T} = 0.5\,\text{J}$. What is the mean deflection $\overline{x}$ and the mean quadratic deflection $\overline{x^2}$?

6.4 Outlook on general relativity

The fundamental idea of *general relativity* is the *equivalence* of inert and heavy mass. While special relativity follows from Lorentz invariance, general relativity follows from Lorentz boost invariance, see also Secs. ?? and ??.

Example 30 (*Relativistic correction to Newton's law*): .

6.4.1 Gravitational red-shift

The *gravitational red-shift* $\Delta\omega$ suffered by a clock of mass m can be estimated from (see Sec.??),

$$\boxed{\hbar\Delta\omega = m\Delta\frac{V(\mathbf{r})}{m}}\,, \tag{6.45}$$

where $\Delta V(\mathbf{r})$ is the gravitational potential difference with and without a nearby heavy mass. The mass of the clock is a measure of its pace: $m = E/c^2 = \hbar\omega/c^2$. For instance, on the surface of Earth we get,

$$\hbar\Delta\omega = mg\Delta z = \frac{E}{c^2}g\Delta z = \frac{\hbar\omega}{c^2}g\Delta z \ . \tag{6.46}$$

Hence,

$$\frac{\Delta\omega}{\omega} = \frac{g}{c^2}\Delta z \simeq \Delta z \cdot 10^{-16}\,\mathrm{m}^{-1} \ . \tag{6.47}$$

6.4.2 Exercises

6.5 Further reading

H.M. Nussenzveig, Edgar Blucher (2013), *Curso de Física Básica: Mecânica - vol 1* [ISBN]

Chapter 7

Hydrodynamics

This chapter is not elaborated, yet. It is mainly included because of some nice exercises adapted from [7].

7.1 Hydrostatics

7.1.1 Buoyancy and Archimedes' principle

The density of a substance is the quotient between its mass and its volume,

$$\rho \equiv \frac{m}{V} . \tag{7.1}$$

Archimedes' principle says that

> A body immersed in a liquid undergoes the action of an upward force, whose absolute value is equal to the weight of the volume of liquid displaced by the body.

Let us consider a liquid, with density ρ_{lq}, in hydrostatic equilibrium inside a container and highlight a portion of it, with volume V, as shown in Fig. 7.1(a). In order to have hydrostatic equilibrium, it is necessary that the sum over all forces acting on the highlighted volume of liquid be zero. One of these forces is the weight,

$$\mathbf{G} = m\mathbf{g} = \rho V \mathbf{g} , \tag{7.2}$$

of the volume V. The other force is the sum $\mathbf{E}$ over all pressure forces that the remaining liquid exerts on the surface of the volume V, Fig. 7.1(b). That is,

$$\mathbf{G} + \mathbf{E} = 0 . \tag{7.3}$$

Thus, the force $\mathbf{E}$ which 'pushes' the highlighted portion of liquid, has a magnitude equal to its weight, $E = G = \rho_{\text{lq}} V g$, and is called *buoyancy*.

In case that the highlighted volume V is filled with another body with density ρ_{bd} different from that of the liquid ρ_{lq}, the buoyancy will not change. That is, the buoyancy E will always be the weight of the liquid displaced by the body that has been introduced in its interior.

On the other hand, the new weight of the volume is $G_{\text{bd}} = \rho_{\text{bd}} m V$. In the case $\rho_{\text{lq}} > \rho_{\text{bd}}$, the body submerged in the liquid will rise to the surface, since the buoyancy

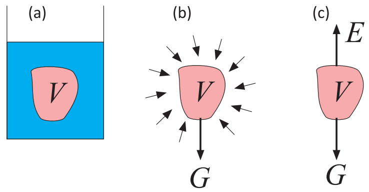

Figure 7.1: Representation of the forces acting on a body submerged in a liquid.

exerted by the liquid will be greater than the weight of the body. Otherwise, when $\rho_{\text{lq}} < \rho_{\text{bd}}$, the submerged body will sink to the bottom of the container. In both cases, the submerged body will not stay in hydrostatic equilibrium. Resolve the Excs. 7.1.2.1 to 7.1.2.8.

Example 31 (*Density Measurement exploiting Archimedes' principle*): We can exploit Archimedes' principle to determine the density of a material of unknown composition. To this end, we first measure the weight of the body in air,

$$G_{\text{air}} = \rho_{\text{bd}} V g \ . \tag{7.4}$$

We then fully immerse the body into a liquid of known density ρ_{lq} (assuming the body to be denser than that of a liquid), and measure the new weight,

$$G_{\text{lq}} = G_{\text{air}} - E = \rho_{\text{bd}} V g - \rho_{\text{lq}} V g \ . \tag{7.5}$$

The density of the body's material is then,

$$\rho_{\text{bd}} = \frac{\rho_{\text{lq}}}{1 - \frac{G_{\text{lq}}}{G_{\text{air}}}} \ . \tag{7.6}$$

7.1.2 Exercises

7.1.2.1 Ex: Critical buoyancy and Cartesian diver

A balloon is filled at atmospheric pressure with 1 liter of air. An aluminum weight of 1 kg is attached to it. The whole setup is now immersed into water. Calculate the total force acting on the setup as function of the depth of immersion.

7.1.2.2 Ex: Buoyancy of a wooden ball in a liquid

A wooden ball of mass m is trapped at depth h in a liquid of density ρ_{lq}. Having released the ball from its position, determine how high above the surface it will float.

7.1.2.3 Ex: Buoyancy of oil

Three containers with false bottoms (see Fig. 7.2) were placed in water at the same depth. Filling into the three bottles the same amount of oil, which one of the three bottoms will fall first? Justify!

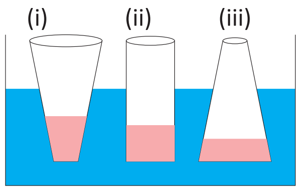

Figure 7.2: Buoyancy of oil.

7.1.2.4 Ex: A tank

A lidless rectangular tank, with the dimensions given in the figure, moves with an acceleration a and contains water up to a height of h (when $a = 0$). At what value of the acceleration will the water begin to flow out?

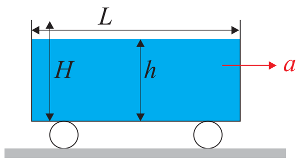

Figure 7.3: Scheme of the tank.

7.1.2.5 Ex: Buoyancy in mercury and water

A cube of a certain material floats in a container containing mercury ($\rho_{\mathrm{Hg}} = 13.6\,\mathrm{g/cm^3}$) such that $1/4$ of its volume is submerged. Adding enough water to the system to cover the cube, what fraction of its volume will still remain immersed in mercury?

7.1.2.6 Ex: Immersion

A board of length L leans on a stone and is partially immersed in water. As the figure shows, a portion of length a is above the support point. Being ρ_{wd} is the density of the wood, what portion of the board is submerged?

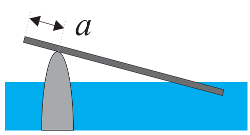

Figure 7.4: Immersion of a stone.

7.1.2.7 Ex: Conical container

Milk is placed in a conical container. Over time, occurs at the top the formation of cream, which is less dense. During this process there is no change in volume, that is, h remains constant (see Fig. 7.5). What happens to the pressure at the bottom of the container? Justify your answer.

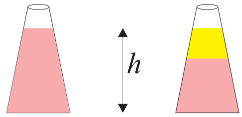

Figure 7.5: Conical container.

7.1.2.8 Ex: Rotating pipe

A water pipe rotates at a speed ω around a vertical axis, as shown in Fig. 7.6. Calculate the pressure as a function of r, using $P(r=0) = P_0$.

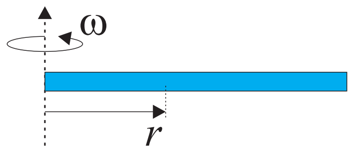

Figure 7.6: Water pipe.

7.2 Hydrodynamics

Resolve the Excs. 7.2.2.1 to 7.2.2.4.

7.2.1 Continuity and Navier-Stokes equations

7.2.1.1 Bernoulli's theorem

7.2.1.2 Viscosity

7.2.2 Exercises

7.2.2.1 Ex: Cylindrical can

a. Into a cylindrical can of area A water is poured up to a height of h. Determine the velocity v at which water flows out of a hole of area A_{hole} located at the bottom. How much water should be added to the can per unit of time such that v is constant?
b. In case that no water is added to the can and the height varies, calculate the flow rate θ as a function of time.

7.2.2.2 Ex: Siphon

Via a siphon, water is removed from a container, as shown in the figure. The area of the pipe is constant along its length and the velocity of the surface of the liquid is neglected.
a. What is the water speed at the outlet of the pipe?
b. What is the pressure at the highest point of the siphon?
c. What is the maximum height h at which it is still possible to siphon water?

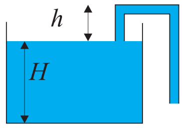

Figure 7.7: Siphon.

7.2.2.3 Ex: Water tank

A water tank is mounted on a wagon that can move horizontally without friction (see Fig. 7.8). In the wall of the box there is a hole of area A at a depth H, through which water flows parallel to the horizontal plane. The initial total mass of the system (box, water and wagon) is M_0 and the speed of the water surface is neglected. If the wagon is initially at rest when the hole is opened, what will be the initial acceleration of the system?

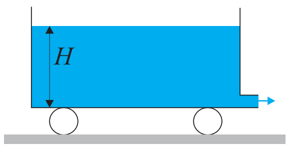

Figure 7.8: Water tank.

7.2.2.4 Ex: Flow meter

A flow meter consists of a conical, vertical glass tube with a metallic sphere of mass m and radius r fixed inside it, as shown in the figure. Calculate the flow of a gas with a given viscosity η as a function of the height h. Consider, very small α. **Note:** $F_{\text{Stokes}} = 6\pi\eta r v$.

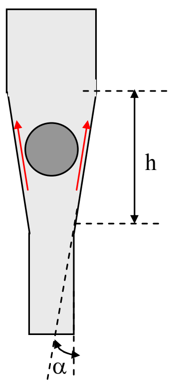

Figure 7.9: Flow meter.

7.3 Further reading

J. Pedlosky (1992), *Geophysical Fluid Dynamics* [ISBN]

Chapter 8

Appendices to 'Classical Mechanics'

8.1 Constants and units in classical physics

8.1.1 Constants

8.1.1.1 Mathematical constants

π constant	$\pi = 3.1415..$
Euler constant	$e = 2.71828..$

8.1.1.2 Constants of the SI unit system

These numbers of the special adjustment CODATA 2019 were proposed as *exact* values.

frequency of the hyperfine transition of Cs	$\nu = 9\ 192\ 631\ 770\,\mathrm{Hz}$
velocity of light	$c = 299\ 792\ 458\,\mathrm{m/s}$
Planck's constant	$h = 6.626\ 070\ 15 \times 10^{-34}\,\mathrm{Js}$
electronic charge	$e = 1.602\ 176\ 634 \times 10^{-19}\,\mathrm{C}$
Boltzmann's constant	$k_B = 1.380\ 649 \times 10^{-23}\,\mathrm{J/K}$
Avogadro's constant	$N_A = 6.022\ 14076 \times 10^{23}\,\mathrm{mol^{-1}}$
Luminous efficiency	$K_{cd} = 683\,\mathrm{lm}$

The main fundamental constants are each one representative for a ground braking physical theory: c for special relativity, e for electrodynamics, h for quantum mechanics, and k_B for statistical thermodynamics.

8.1.1.3 Derived constants

fine-structure constant	$\alpha = e^2/4\pi\varepsilon_0\hbar c \approx 1/137$
vacuum permittivity	$\varepsilon_0 = 1/\mu_0 c^2 = 8.8542 \times 10^{-12}\,\text{As/Vm}$
vacuum permeability	$\mu_0 = 10^{-7}\,\text{Vs/Am}$
Faraday's constant	$F = 96485.309\,\text{C/mol}$
atomic mass unit	$u_A = 1/N_A \times 1\text{g/mol} = 1.6605402 \times 10^{-27}\,\text{kg}$
gas constant	$R = N_A k_B = 8.314510\,\text{L/mol K}$
Bohr radius	$a_B = \alpha/4\pi R_\infty = 0.529 \times 10^{-10}\,\text{m}$
Bohr magneton	$\mu_B = e\hbar/2m_e = 9.27 \times 10^{-24}\,\text{J/T}$
classical electron radius	$r_e = \alpha^2 a_B$
Rydberg constant	$R_\infty = m_e c\alpha^2/2h = 13.7\,\text{eV}$
Compton wavelength	$\lambda_C = h/m_e c$
Thomson cross section	$\sigma_e = (8\pi/3)r_e^2$
gravitational constant	$\gamma = 6.67259 \times 10^{-11}\,\text{m}^3\text{kg}^{-1}\text{s}^{-2}$

8.1.1.4 Particle constants

electron mass	$m_e = 9.1096 \times 10^{-12}\,\text{kg}$
g-factor of the electron	$g = 2.002\ 319\ 304\ 386$
muon mass	$m_\mu = 105.658389\,\text{MeV}$
proton mass	$m_p = 938.27231\,\text{MeV}$
g-factor of the proton	$g = 5.5858$
neutron mass	$m_p = 939.56563\,\text{MeV}$
g-factor of the neutron	$g = -3.8261$
deuteron mass	$m_d = 1875.61339\,\text{MeV}$

8.1.1.5 Astronomical constants

earth mass	$m_{♁} = 5.9736 \times 10^{24}\,\mathrm{kg}$
earth radius	$R_{♁} = 6370\,\mathrm{km}$
earth gravity	$g_{♁} = 9.80665\,\mathrm{m/s}$
lunar mass	$m_{☾} = 7.348 \times 10^{22}\,\mathrm{kg}$
lunar radius	$R_{☾} = 1740\,\mathrm{km}$
lunar gravity	$g_{☾} = 1.62\,\mathrm{m/s}$
distance earth-moon	$d_{ES} = 384000\,\mathrm{km}$
sun massa	$m_{☼} = 1.99 \times 10^{30}\,\mathrm{kg}$
sun radius	$R_{☼} = 695300\,\mathrm{km}$
sun gravity	$g_{☼} = 273\,\mathrm{m/s}$
distance earth-sun	$d_{ES} = 1.496 \times 10^{8}\,\mathrm{km}$
sinodic day	$d_{syn} = 24\,\mathrm{h}$
sideric day	$d_{syn} = 23.9345\,\mathrm{h} = 23\,\mathrm{h}\,56\,\mathrm{min}\,4\,\mathrm{s}$
sinodic month	$mon_{syn} = 29.530590\,\mathrm{d}$
sideric month	$mon_{sid} = 27.321666\,\mathrm{d}$
sideric year	$a_{syn} = 365.256365\,\mathrm{h} = 365\,\mathrm{d}\,6\,\mathrm{h}\,9\,\mathrm{min}\,10\,\mathrm{s}$
lunar day	$d_{lunar} = 24.8412\,\mathrm{h}$
$\frac{1}{mon_{sid}}$	$= \frac{1}{a_{sid}} + \frac{1}{mon_{syn}}$
$\frac{1}{d_{sid}}$	$= \frac{1}{a_{sid}} + \frac{1}{d_{syn}}$
$\frac{1}{d_{sid}}$	$= \frac{1}{mon_{sid}} + \frac{1}{d_{lunar}}$

8.1.2 Units

charge	Q	basic unit
current	I	A=C/s
voltage	U	V=N/As
polarizability	α_{pol}	Asm^2/V
susceptibility	χ	1
dipolar moment	1 Debye	$= 10^{-27}/2.998$ Cm $= 10^{-19}/c$ Cm^2/s $= 39.36\ ea_B$

8.2 Quantities and formulas in classical mechanics

time	t	basic unit
position	$\mathbf{r}$	basic unit
velocity	$\mathbf{v}$	$\mathbf{v} = \dot{\mathbf{r}}$
acceleration	$\mathbf{a}$	$\mathbf{a} = \dot{\mathbf{v}}$
mass	m	basic unit
linear momentum	$\mathbf{p}$	$\mathbf{p} = m\mathbf{v}$
force	$\mathbf{F}$	$\mathbf{F} = \dot{\mathbf{p}} = m\mathbf{a}$
kinetic energy	E_{kin}	$E_{kin} = \frac{m}{2}v^2$
angle	$\vec{\phi}$	basic unit
angular velocity	$\vec{\omega}$	$\vec{\omega} = \dot{\mathbf{r}}$
angular acceleration	$\vec{\alpha}$	$\vec{\alpha} = \dot{\vec{\omega}}$
inertial moment (continuous density)	I	$I = \int r_\perp^2 dm = \int_V \rho(\mathbf{r})[\mathbf{r}^2 - (\mathbf{r} \cdot \hat{\mathbf{e}}_\omega)]dV$
inertial moment (discrete density)	I	$I = \sum_i m_i r_i^2$
angular momentum	$\mathbf{L}$	$\mathbf{L} = I\vec{\omega} = \mathbf{r} \times \mathbf{p}$
torque	$\vec{\tau}$	$\vec{\tau} = \dot{\mathbf{L}} = I\vec{\alpha} = \mathbf{r} \times \mathbf{F}$
rotational energy	E_{rot}	$E_{rot} = \frac{m}{2}\omega^2 r^2$
potential energy	E_{pot}	$E_{pot}^{grav} = mgh$, $E_{pot}^{sprng} = \frac{k}{2}x^2$
work	W	$W = \int_{s_1}^{s_2} \mathbf{F} \cdot d\mathbf{s}$
power	P	$P = \dot{W}$

8.2.1 Particular forces

gravitation	$F_{grav} = mg$
Hooke's for elastic spring	$F_{mola} = -k\Delta x$
friction	$F_{at} = -\mu N$
Stokes' friction	$F_{fr} = -\gamma v$
Newton's friction	$F_{fr} = -\gamma v^2$

8.2.2 Inertial momentum

Steiner's theorem	$I_{\omega_2} = I_{\omega_1} + md^2$, where d is the distance between parallel axes
theorem of perpendicular axes	$I_z = I_x + I_y$ para $\rho(\mathbf{r}) = \delta(z)\sigma(x, y)$

8.2.3 Inertial forces due to transitions to translated and rotated systems

transformation to an accelerated frame	$\mathbf{F}_{Gal} = -m\mathbf{a}$
centrifugal force	$\mathbf{F}_{cf} = -m\vec{\omega} \times (\vec{\omega} \times \mathbf{r})$
Coriolis force	$\mathbf{F}_{Cor} = -2m\vec{\omega} \times \mathbf{v}$

8.2.4 Conservation laws

energy conservation	$\sum_k E_{kin}^{(ini)} + \sum_k E_{pot}^{(ini)} = \sum_k E_{kin}^{(fin)} + \sum_k E_{pot}^{(fin)}$
linear momentum conservation	$\sum_k \mathbf{p}_k^{(ini)} = \sum_k \mathbf{p}_k^{(fin)}$
angular momentum conservation	$\sum_k \mathbf{L}_k^{(ini)} = \sum_k \mathbf{L}_k^{(fin)}$
definition of the center-of-mass	$\mathbf{r}_{cm} \equiv \frac{\sum_k m_k \mathbf{r}_k}{\sum_k m_k}$

8.2.5 Rigid bodies, minimum required number of equations of motion

1. estimate number of moving masses	m_1, m_2, ...
2. identify possible movement (degree of freedom) for every mass	$v_{1x}, v_{2x}, \ldots$ $v_{1y}, v_{2y}, \ldots$ $v_{1z}, v_{2z}, \ldots$ $\omega_1, \omega_2, \ldots$
3. write down for every degree of freedom an equation of motion	$m\dot{v}_{kl} = \sum_j F_j$ $I\dot{\omega}_k = \sum_j \tau_j$

8.2.6 Gravitational laws

Newton's law	$\mathbf{F}(\mathbf{r}) = -\frac{GMm}{\|\mathbf{R}-\mathbf{r}\|^2}\hat{\mathbf{e}}_{Rr} = -\nabla V(\mathbf{r})$
gravitational potential	$V(\mathbf{r}) = -\int \frac{Gm}{\|\mathbf{r}-\mathbf{r}'\|^2}\rho(\mathbf{r}')dV'$

8.2.7 Volume elements

cartesian coordinates	$dV = dxdydz$
cylindrical coordinates	$dV = \rho d\rho d\phi dz$
spherical coordinates	$dV = r^2 \sin\theta dr d\theta d\phi$

8.2.8 Oscillations $ma + bv + kx = F_0 \cos\omega t$

dissipative motion	$k = 0,\ F_0 = 0$	$x(t) = Ae^{-\gamma t}$, $\gamma = \frac{b}{2m}$
harmonic oscillation	$b = 0,\ F_0 = 0$	$x(t) = A\cos(\omega_0 t + \delta)$, $\omega_0 = \sqrt{\frac{k}{m}}$
damped oscillation	$F_0 = 0$	$x(t) = Ae^{-\gamma t}\cos(\omega t + \delta)$, $\omega = \sqrt{\omega_0^2 - \gamma^2}$
forced oscillation		$x(t) = A\cos(\omega t + \delta)$, $A = \frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2+b^2\omega^2}}$, $\tan\delta = \frac{b\omega}{m(\omega_0^2-\omega^2)}$

8.3 Probability distributions

8.3.1 Binomial distribution

The *binomial distribution* is defined by,

$$B_k^{(n)} = \binom{n}{k} p^k (1-p)^{n-k} \ . \tag{8.1}$$

It normalized,

$$1 = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \ , \tag{8.2}$$

and,

$$\begin{aligned} \int_0^1 B_k^{(n)}(p)dp &= \binom{n}{k} \int_0^1 p^k (1-p)^{n-k} dp \\ &= \binom{n}{k+1} \int_0^1 p^{k+1}(1-p)^{n-k-1} dp = \int_0^1 p^n dp = \frac{1}{n+1} \ . \end{aligned} \tag{8.3}$$

8.3.2 Poisson distribution

The *Poisson distribution* is defined by,

$$P_k = \frac{\lambda^k}{k!} e^{-\lambda} \ . \tag{8.4}$$

For large n (typically $n > 100$) and small p (typically $p < 10$), we get $B_k^{(n)} \simeq P_k$ with $\lambda = np$. To see this, we use two approximation formulas. The first formula,

$$\log(1-p) \simeq -p \tag{8.5}$$

immediately follows a Taylor expansion. The second formula,

$$\frac{N!}{(N-n)!} \simeq N^n \tag{8.6}$$

follows from,

$$\frac{N!}{(N-n)!} \equiv \prod_{j=0}^{n-1}(N-j) \overset{n \ll N}{\simeq} \prod_{j=0}^{n-1} N \equiv N^n \ . \tag{8.7}$$

Hence,

$$\begin{aligned}(1-p)^{N-n} &= \exp\left(log(1-p)^{N-n}\right) \\ &= \exp\left(N-n\right)\log\left(1-p\right) \simeq \exp\left(-N\,p\right) = \exp\left(-\alpha\right) \ .\end{aligned} \tag{8.8}$$

Finally,

$$\frac{N!}{(N-n)!}\,p^n \simeq N^n\,p^n = \alpha^n \ . \tag{8.9}$$

8.3.3 Gaussian distribution

The *Gaussian distribution* is defined by,

$$G_k = \frac{1}{2\pi\sigma^2} e^{-(k-\lambda)^2/2\sigma^2} \ . \tag{8.10}$$

For large n, we get $B_k^{(n)} \simeq G_k$ with $\sigma = \sqrt{np(1-p)}$.

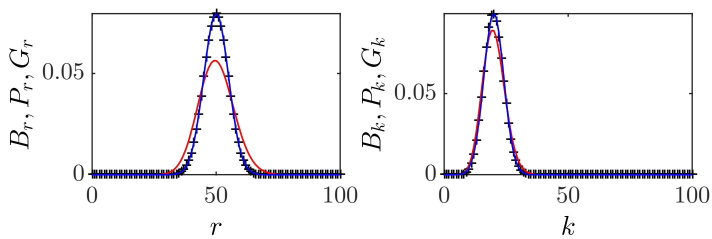

Figure 8.1: (code) (black) Binomial, (red) Poissonian, and (blue) Gaussian probability distribution for $N = 100$ and (left) $Np = 50$ and (right) $Np = 20$.

8.3.4 Some useful formulae

If the limits of two functions tend to 0, $\lim_{t\to t_0} f(t) = 0 = \lim_{t\to t_0} g(t)$ a rule called *l'Hôpital's rule* goes like,

$$\lim_{t\to t_0} \frac{f(t)}{g(t)} = \lim_{t\to t_0} \frac{f'(t)}{g'(t)} \ . \tag{8.11}$$

Bibliography

[1] H. M. Nussenzveig, *Curso de física básica: Mecânica - vol 1*, 7 ed., Edgar Blucher, 2013, ISBN.

[2] A. E. Siegman, *Lasers*, 1986, ISBN.

[3] C. F. von Weizsäcker, *Einheit der natur*, München, 1971, ISBN.

[4] ———, *Aufbau der physik*, München, 2006, ISBN.

[5] J. Weiner and P.-T. Ho, *Light-matter interaction, fundamentals and applications*, John Wiley & Sons, Hokoken, New Jersey, 2003, ISBN.

[6] N. Wermuth, M. Wiedenheck, and D. R. Cox, *Partial inversion for linear systems and partial closure of independent graphs*, BIT Numer. Math. **46** (2006), 883, DOI.

[7] S. C. Zilio and V. S. Bagnato, *Mecânica, calor, ondas*, script, 2000, .

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