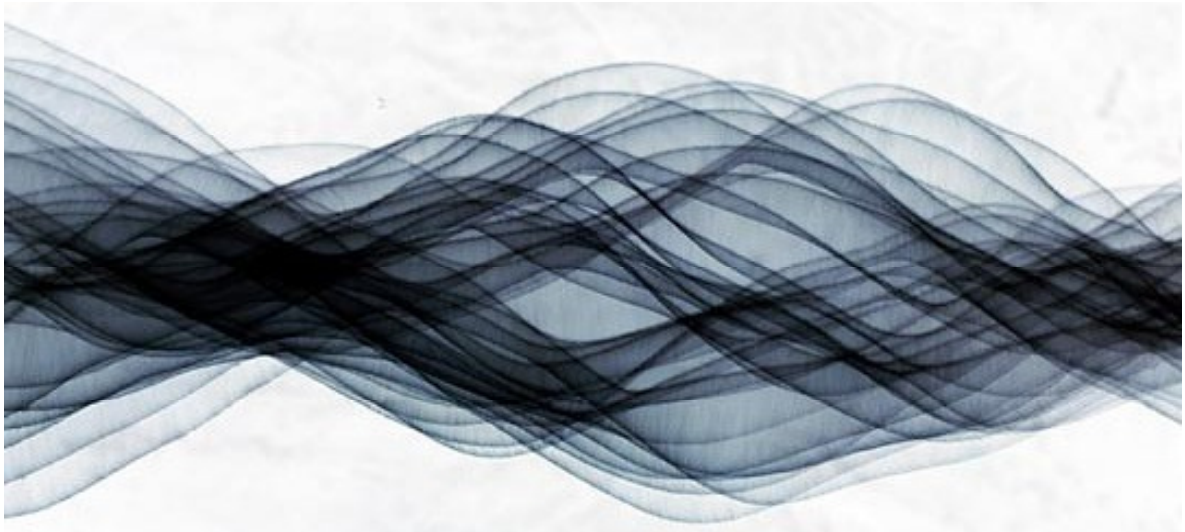

Undergraduate courses in Physics

Classical Mechanics

Vibrations, Waves and Gravity

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Preface

Script with subjects and exercises for graduation in Physics relevant to the courses of the IFSC: FCM0501 (Física I para físicos), FCM0101 (Física I para engenheiros e matemáticos), FFI0405 (Física Geral I para engenheiros e matemáticos), FCM0200 (Física Básica I para engenheiros e matemáticos).

Information and announcements regarding the course will be published on the website:
http://www.ifsc.usp.br/ strontium/ $-$ > Teaching $-$ > FFI0132

The following literature is recommended for preparation and further reading:

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Content

Part I

Classical Mechanics

Chapter 1

Vibrations

Vibrations are periodic processes, that is, processes that repeat themselves after a given time interval. After a time called *period*, the system under consideration returns to the same state in which it was initially. There innumerable examples for periodic processes, such as the motion of a seesaw, oceanic tides, electronic $L - C$ circuits, alternating current or rotations like that of the Earth around the Sun. Thus, vibrations are among the most fundamental processes in all domains of physics. A lecture version of this chapter can be found at (watch talk).

1.1 Free periodic motion

A movement is considered as free, when apart from a *restoring force*, that is a force working to counteract the displacement, there are no other forces accelerating or slowing down the motion.

1.1.1 Clocks

Periodic motions are used to *measure time*. Assuming a given process to be truly periodic, we can inversely *postulate* that the time interval within which this process occurs is constant. This interval is used to define a *unit of time*. For example, the 'day' is defined as the interval that the Earth needs to complete a rotation about its axis. The 'second' is defined as the 86400-th fraction of this period. Taking the second inversely as the base unit, we can define the day as the time interval needed for a periodic process taking 1 s to occur 86400 times. That is, we count the number of times ν that this process occurs within a day and calculate the duration of a day through,

$$\Delta T = \frac{1}{\nu} \ . \tag{1.1}$$

In real life, vibrations are subject to perturbations, just like all physical processes. These perturbations may afflict the periodicity and falsify the measurement of time. For example, the oceanic tides, which depend on the rotation of the moon around the Earth, can influence the Earth's own rotation. One of the challenges of *metrology*, which is the science dealing with issues related to the measurement of time, is to identify processes in nature that are likely to be insensitive to external perturbations. Nowadays, the most stable known periodic processes are vibrations of electrons within atoms. Therefore, the international time is defined by an atomic clock based on

cesium: The 'official' second is the time interval in which the state of an electron oscillates 9192631770 times when the hyperfine structure of a cesium atom is excited by a microwave.

The unit of time is,

$$\text{unit}(T) = \text{s} \ . \tag{1.2}$$

A *frequency* is defined as the number of processes that occur within one second. We use the unit,

$$\text{unit}(\nu) = \text{Hz} \ . \tag{1.3}$$

Often, to simplify mathematical formulas, we will use the derived quantity of the *angular frequency* also called *angular velocity*,

$$\omega \equiv 2\pi\nu \ . \tag{1.4}$$

It has the unit,

$$\text{unit}(\omega) = \text{rad/s} \neq \text{Hz} \ . \tag{1.5}$$

It is important not to use the unit 'Hertz' for angular frequencies in order to avoid confusion.

1.1.2 Periodic trajectories

Many periodic processes are based on repetitive trajectories of particles or bodies. As an example, let us the movement of a body in a box shown in Fig. 1.1. When the body encounters a wall, it is elastically reflected thereby maintaining its velocity but reversing the direction of propagation. Clearly, the velocity is the derivative of the position,

$$v(t) = \dot{x}(t) \ . \tag{1.6}$$

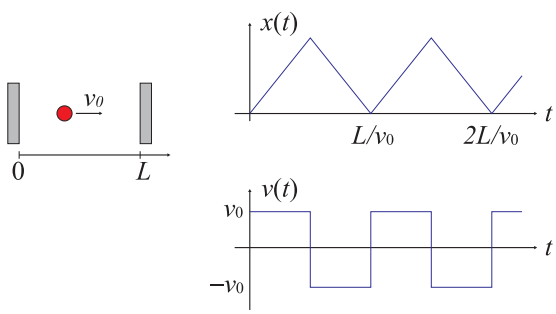

Figure 1.1: Trajectory of a body in a rectangular box. Upper trace: instantaneous position. Lower trace: instantaneous velocity.

To fully describe the trajectory of a body and to identify, when the trajectory repeats, two parameters are needed. Specifying, for example, the time evolution of position $x(t)$ and velocity $v(t)$, we can search for time intervals T after which,

$$x(t_0 + T) = x(t_0) \qquad \text{and} \qquad v(t_0 + T) = v(t_0) \ . \tag{1.7}$$

Obviously, as seen in Fig. 1.1, it is not enough just to look for the time when $x(t_0 + T) = x(t_0)$.

1.1.3 Simple harmonic motion

The simplest motion imaginable is the harmonic oscillation described by,

$$x(t) = A\cos(\omega_0 t - \phi) \ , \tag{1.8}$$

and exhibit in Fig. 1.2. A is the *amplitude* of the motion, such that $2A$ is the distance between the two turning points. $T = 2\pi/\omega_0$ is the oscillation period, since,

$$\cos[\omega_0(t+T) - \phi] = \cos[\omega_0 t + 2\pi - \phi] = \cos[\omega_0 t - \phi] \ . \tag{1.9}$$

ϕ is a *phase shift* describing the time delay $t = \phi/\omega_0$ for the oscillation to reach the turning point.

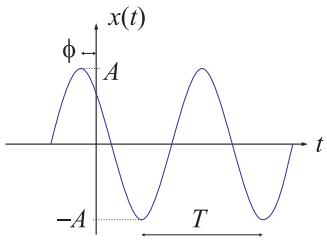

Figure 1.2: Illustration of the cosenus function with the amplitude A, the period T and the phase being negative for this graph $\phi < 0$.

The velocity and acceleration follow from,

$$v(t) = \dot{x}(t) = -\omega_0 A\sin(\omega_0 t - \phi) \qquad \text{and} \qquad a(t) = \dot{v}(t) = -\omega_0^2 A\cos(\omega_0 t - \phi) \ . \tag{1.10}$$

with this we can, using Newton's law, calculate the force necessary to sustain the oscillation of the body,

$$F(t) = ma(t) = -m\omega_0^2 A\cos(\omega_0 t - \phi) = -m\omega_0^2 x(t) \equiv kx(t) \ . \tag{1.11}$$

That is, in the presence of a force, which is proportional to the displacement but with the opposite direction, $F \propto -x$, we expect a sinusoidal solution. The proportionality constant k is called *spring constant*. Obviously the oscillation frequency is independent of amplitude and phase,

$$\omega_0 = \sqrt{k/m} \ . \tag{1.12}$$

Solve Exc. 1.1.10.1.

Example 1 (*Harmonic vibration*):

- Suspended spring-mass system, pendulums with various masses and lengths of wire, oscilloscope and function generator, water recipient with a floating body.

1.1.4 The spring-mass system

Let us now discuss a possible experimental realization of a sinusoidal vibration. Fig. 1.3 illustrates the spring-mass system consisting of a mass horizontally fixed to a spring. This system has a resting position, which we can set to the point $x = 0$, where no forces act on the mass. When elongated or compressed, the spring exerts a restoring force on the mass working to bring the mass back into its resting position,

$$F_{restore} = -kx \ . \tag{1.13}$$

This so-called *Hooke's law* holds for reasonably small elongations. The spring coefficient k is a characteristic of the spring.

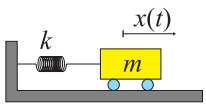

Figure 1.3: Illustration of the spring-mass system.

The oscillation frequency of the spring-mass system is determined by the spring coefficient and the mass, but the phase and the amplitude of the oscillation are parameters, that depend on the way the spring-mass is excited. Knowing the position and velocity of the oscillation at a given time, that is, the initial conditions of the motion, we can determine the amplitude and phase. To see this, we expand the general formula for a sinusoidal oscillation,

$$x(t) = A\cos(\omega_0 t - \phi) = A\cos(\omega_0 t)\cos\phi + A\sin(\omega_0 t)\sin\phi \tag{1.14}$$

and calculate the derivative,

$$v(t) = -A\omega_0\cos\phi\sin(\omega_0 t) + A\omega_0\sin\phi\cos(\omega_0 t) \ . \tag{1.15}$$

With the initial conditions $x(0) = x_0$ and $v(0) = v_0$ we get,

$$A\cos\phi = x_0 \qquad \text{and} \qquad A\omega_0\sin\phi = v_0 \ . \tag{1.16}$$

Hence,

$$x(t) = x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t) \ . \tag{1.17}$$

Solve the Excs. 1.1.10.2, 1.1.10.3, 1.1.10.4, and 1.1.10.5.

1.1.5 Energy conservation

Considerations of *energy conservation* can often help solving mechanical problems. The kinetic energy due to the movement of the mass m is,

$$E_{kin} = \tfrac{m}{2}v^2 \ , \tag{1.18}$$

and the potential energy due to the restoring force is,

$$E_{pot} = -\int_0^x F dx' = -\int_0^x -kx' dx' = \tfrac{k}{2}x^2 \ . \tag{1.19}$$

The total energy must be conserved:

$$E = E_{kin} + E_{pot} = \tfrac{m}{2}v^2 + \tfrac{k}{2}x^2 = const \ , \tag{1.20}$$

but is continuously transformed between kinetic energy and potential energy. This is illustrated on the left-hand side of the Fig. 1.4.

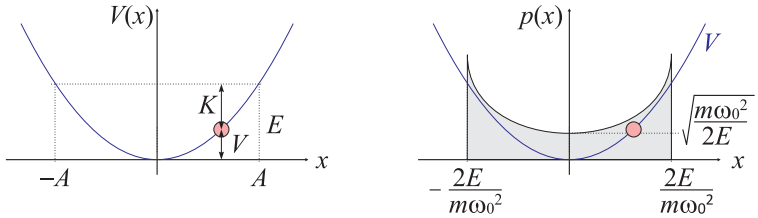

Figure 1.4: (Left) Energy conservation in the spring-mass system showing the kinetic energy K, the potential energy V, and the total energy E. (Right) Probability density of finding the oscillator in position x.

Example 2 (*Probability distribution in the harmonic oscillator*): Let us now use the principle of energy conservation to calculate the probability of finding the oscillating mass next to a given displacement x. For this, we solve the last equation by the velocity,

$$v = \frac{dx}{dt} = \sqrt{\frac{2}{m}E - \tfrac{k}{m}x^2} = \omega_0\sqrt{\frac{2E}{m\omega_0^2} - x^2} \ , \tag{1.21}$$

or

$$\frac{dx}{\sqrt{\frac{2E}{m\omega_0^2} - x^2}} = \omega_0 dt \ . \tag{1.22}$$

The probability of finding the mass within a given time interval dt is,

$$p(t)dt = \frac{dt}{T} = \frac{\omega_0}{2\pi}dt = \frac{dx}{2\pi\sqrt{\frac{2E}{m\omega_0^2} - x^2}} = \tilde{p}(x)dx \ . \tag{1.23}$$

Hence,

$$\tilde{p}(x) = \frac{1}{2\pi\sqrt{\frac{2E}{m\omega_0^2} - x^2}} \tag{1.24}$$

is the probability density of finding in the mass at the position $x(t)$. Using $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}$ with $x_0 = \sqrt{\frac{2E}{m\omega_0^2}}$ we verify,

$$2\int_{-x_0}^{x_0} \tilde{p}(x)dx = \frac{1}{\pi}\left[\arcsin\frac{x}{\sqrt{\frac{2E}{m\omega_0^2}}}\right]_{-x_0}^{x_0} = \frac{2}{\pi}\arcsin\frac{x_0}{\sqrt{\frac{2E}{m\omega_0^2}}} = \frac{2}{\pi}\arcsin 1 = 1 \ . \tag{1.25}$$

The probability density is shown on the right side of Fig. 1.4 [1].

[1] To understand the difference between the probability densities $p(t)$ and $\tilde{p}(x)$ we imagine the following experiments: We divide the period T into equal intervals dt and take a series of photos, all with the same exposure time dt. To understand the meaning of $p(t)$, we throw a random number to choose one of the photos. Each photo has the same probability dt/T to be chosen and, of course, $\int_0^T p(t)dt = 1$. To understand the meaning of $\tilde{p}(x)$, we identify the position of the oscillator in each photo and plot it in a histogram. This histogram is reproduced by $\tilde{p}(x)$.

1.1.6 The spring-mass system with gravity

When a mass is suspended vertically to a spring, as shown on the left-hand side of Fig. 1.5, the gravitational force acts on the mass in addition to the restoring force. This can be expressed by the following balance of forces,

$$ma = -ky - mg \ , \tag{1.26}$$

letting the y-axis be positive in the direction opposite to gravitation. Replacing $\tilde{y}' \equiv y - y_0$ with $y_0 \equiv -\frac{mg}{k}$, we obtain,

$$m\tilde{a} = -k\tilde{y} \ . \tag{1.27}$$

Therefore, the movement is the same as in the absence of gravitation, but around an equilibrium point shifted downward by y_0.

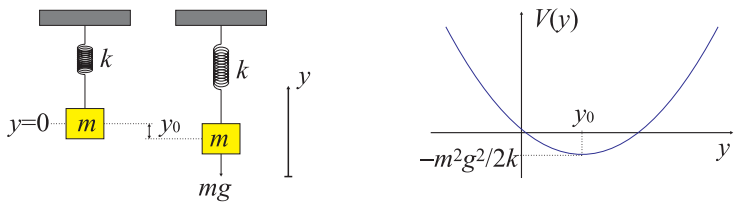

Figure 1.5: Left: Vertical spring-mass system. Right: Conservation of energy in the spring-mass system with gravity.

Energy conservation is now generalized to,

$$E = E_{kin} + E_{mol} + E_{grv} = \tfrac{m}{2}v^2 + \tfrac{k}{2}y^2 + mgy = const \ , \tag{1.28}$$

the potential energy being,

$$\begin{aligned} E_{pot} = E_{mol} + E_{grv} &= \tfrac{k}{2}y^2 + mgy \\ &= \tfrac{k}{2}(y-y_0)^2 + \tfrac{k}{2}2y_0y - \tfrac{k}{2}y_0^2 + mgy = \tfrac{k}{2}(y-y_0)^2 - \frac{m^2g^2}{2k} \ . \end{aligned} \tag{1.29}$$

The right-hand side of Fig. 1.5 illustrates the conservation of energy in the spring-mass system with gravity. See Excs. 1.1.10.6, 1.1.10.7, 1.1.10.8, and 1.1.10.9.

1.1.7 The pendulum

The pendulum is another system which oscillates in the gravitational field. In the following, we will distinguish three different types of pendulums. In the *ideal pendulum* the mass of the oscillating body is all concentrated in one point and the oscillations have small amplitudes. In the *physical pendulum* the mass of the body is distributed over a finite spatial region. And *mathematical pendulum* is a point mass oscillating with a large amplitude and therefore subject to a nonlinear restoring force.

Figure 1.6: Physical pendulum.

1.1.7.1 The ideal pendulum

The *ideal pendulum* is schematized on the left side of Fig. 1.6. As the centrifugal force is compensated for by the traction of the wire supporting the mass, the acceleration force ma is solely due to the perpendicular projection $-mg\sin\theta$ on the wire. For small amplitudes, $\sin\theta \simeq \theta$, such that [2],

$$ma \simeq -mg\theta \ . \tag{1.30}$$

The tangential acceleration is now,

$$a = \dot{v} = \ddot{s} = \frac{d}{dt}\theta L = L\ddot{\theta} \ . \tag{1.31}$$

Thus,

$$\ddot{\theta} + \frac{g}{L}\theta \simeq 0 \ . \tag{1.32}$$

This equation has the same structure as that of the already studied spring-mass system $\ddot{x} + \frac{k}{m}x = 0$. Therefore, we can deduce that the ideal pendulum oscillates with the frequency,

$$\omega_0 = \sqrt{\frac{g}{L}} \ , \tag{1.33}$$

only that the oscillating degree of freedom is an angle rather than a spatial shift. It is interesting to note that the oscillation frequency is independent of the mass. See Exc. 1.1.10.10.

1.1.7.2 The physical pendulum

We consider an irregular body suspended at a point P as schematized on the right-hand side of Fig. 1.6. The center-of-mass be displaced from the suspension point by a distance D. This system represents the *physical pendulum*. Gravitation exerts a torque $\vec{\tau}$ on the center-of-mass,

$$\vec{\tau} = \mathbf{D} \times m\mathbf{g} \qquad \text{with} \qquad \tau = I\ddot{\theta} \ , \tag{1.34}$$

where I is the moment of inertia of the body for rotations about the suspension axis. Like this,

$$I\ddot{\theta} = -Dmg\sin\theta \ . \tag{1.35}$$

[2] The equation of motion can be derived from the Hamiltonian $H = \frac{L_\theta^2}{2ml^2} + mgl\cos\theta$ using $\dot{\theta} = \partial H/\partial \text{Ł}_\theta$ and $\dot{L_\theta} = -\partial H/\partial\theta$, where L_θ is the angular momentum.

Considering once more small angles, $\sin\theta \simeq \theta$, we obtain,

$$\ddot{\theta} + \omega_0^2\theta \simeq 0 \qquad \text{with} \qquad \omega_0 \equiv \sqrt{\frac{Dmg}{I}} \ . \tag{1.36}$$

It is worth mentioning that the inertial moment of a body whose mass is concentrated in a point at a distance D from the suspension point follows *Steiner's law*,

$$I = mD^2 \ . \tag{1.37}$$

With this we recover the expression of the ideal pendulum,

$$\omega_0 = \sqrt{\frac{Dmg}{mD^2}} = \sqrt{\frac{g}{D}} \ . \tag{1.38}$$

1.1.7.3 The mathematical pendulum

The equation describing the mathematical pendulum (see Fig. 1.6) has already been derived but, differently from what we did before, here we will not apply the small angle approximation,

$$\ddot{\theta} = -\frac{g}{L}\sin\theta = -\omega_0^2\sin\theta \ . \tag{1.39}$$

Energy conservation can be formulated as follows:

$$\begin{aligned} 0 = \frac{dE}{dt} &= \frac{d}{dt}(E_{rot} + E_{pot}) = \frac{d}{dt}\frac{I}{2}\dot{\theta}^2 + \frac{d}{dt}mgL(1-\cos\theta) \\ &= \frac{I}{2}2\dot{\theta}\ddot{\theta} + mgL\dot{\theta}\sin\theta \simeq \dot{\theta}(I\ddot{\theta} + mgL\theta) \ . \end{aligned} \tag{1.40}$$

Thus, we obtain the same differential equation,

$$\ddot{\theta} + \frac{mgL}{I}\theta = 0 \ . \tag{1.41}$$

Example 3 (*Simulation of an anharmonic pendulum*): When the anharmonicity is not negligible, it is impossible to solve the differential equation analytically. We must resort to numerical simulations. The simplest procedure is an iteration of the type,

$$\begin{aligned} \theta(t+dt) &= \theta(t) + dt\dot{\theta} = \theta(t) + dt\omega \\ \omega(t+dt) &= \omega(t) + dt\dot{\omega} = \omega(t) - dt\omega_0\sin\theta \ . \end{aligned}$$

Fig. 1.7(a) shows the temporal dephasing of the oscillation caused by the anharmonicity as compared to the harmonic oscillation. Fig. 1.7(b) shows the orbits $\theta(t) \longmapsto \omega(t)$ in the phase space.

1.1.8 The spring-cylinder system

Another example of an oscillating system is shown in Fig. 1.8. The inertial moment of the cylinder is $I = \frac{M}{2}R^2$. The spring exerts the force,

$$F_{mol} = -kx \ . \tag{1.42}$$

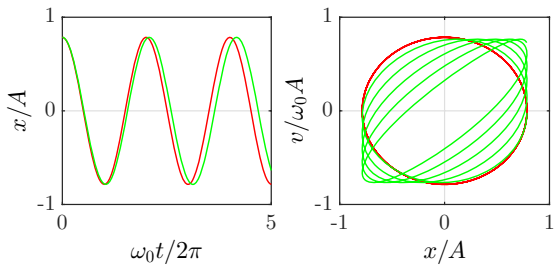

Figure 1.7: (code) Diffusion due to anharmonicities (a) in time and (b) in phase space. The red curves show the harmonic approximation.

Therefore, we have the equations of motion,

$$\begin{aligned} M\ddot{x} &= F_{mol} - F_{at} \\ I\ddot{\theta} &= -RF_{at} \ . \end{aligned} \tag{1.43}$$

If the wheel does not slip, we can eliminate the friction using $x = R\omega$, and we obtain,

$$I\ddot{\theta} = I\frac{\ddot{x}}{R} = \frac{M}{2}R^2\frac{\ddot{x}}{R} = -RF_{at} = -R(-kx - M\ddot{x}) \ . \tag{1.44}$$

Resolving by $\ddot{x}$,

$$\ddot{x} + \frac{2k}{3M}x = 0 \ . \tag{1.45}$$

The frequency is,

$$\omega_0 = \sqrt{\frac{2k}{3M}} \ . \tag{1.46}$$

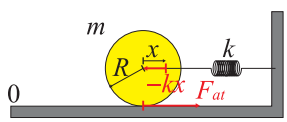

Figure 1.8: The spring-cylinder system.

1.1.9 Two-body oscillation

We now consider the oscillations of two bodies m_1 and m_2 located at the positions x_1 and x_2 and interconnected by a spring k, as shown in Fig. 1.9. The free length, that is, the distance at which the spring exerts no forces on the masses, is ℓ. The forces grow with the stretch $x \equiv x_2 - x_1 - \ell$ of the spring, such that $x > 0$ when the spring is stretched and $x < 0$ when it is compressed. Thereby,

$$m_1\ddot{x}_1 = kx \qquad \text{and} \qquad m_2\ddot{x}_2 = -kx \ . \tag{1.47}$$

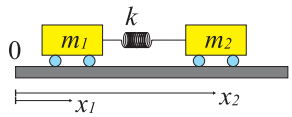

Figure 1.9: Two bodies in relative vibration.

Adding these equations,

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 \equiv (m_1 + m_2)\ddot{x}_{cm} = 0 \ . \tag{1.48}$$

Dividing the equations by the masses and subtracting them,

$$\ddot{x}_1 - \ddot{x}_2 = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)x = \ddot{x}_{rel} = -\frac{k}{\mu}x = \omega_0 x \ , \tag{1.49}$$

where $\omega_0^2 = k/\mu$ and $\mu^{-1} \equiv m_1^{-1} + m_2^{-1}$ is called the *reduced mass*. The introduction of the reduced mass turns the oscillator consisting of two bodies equivalent to a system consisting of only one mass and one spring, but with an increased vibration frequency,

$$\omega_\mu = \sqrt{\frac{k}{\mu}} = \sqrt{2\frac{k}{m}} \ . \tag{1.50}$$

This system represents an important model for the description of *molecular vibration*. Note that for $m_1 \longrightarrow \infty$ we restore the known situation of a spring-mass system fixed to a wall.

1.1.10 Exercises

1.1.10.1 Ex: Zenith in São Carlos

Knowing that the latitude of the Sun in the tropics of Capricorn is $\alpha_{trop} = 23°$ calculate at what time of the year the sun is vertical at noon in São Carlos, SP, Brazil.

1.1.10.2 Ex: Swing modes

In the systems shown in the figure there is no friction between the surfaces of the bodies and floor, and the springs have negligible mass. Find the natural oscillation frequencies.

1.1.10.3 Ex: Coupled springs

A mass m is suspended within a horizontal ring of radius $R = 1\,\mathrm{m}$ by three springs with the constants $D_1 = 0.1\,\mathrm{kg/m}$, $D_2 = 0.2\,\mathrm{N/m}$, and $D_3 = 0.3\,\mathrm{N/m}$. The suspension points of the springs on the ring have the same mutual distances. Determine the equilibrium position of the mass assuming that the springs' extensions at rest range is 0.

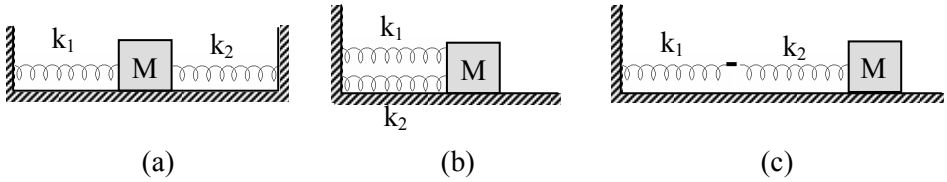

Figure 1.10: Swing modes.

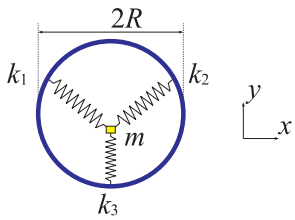

Figure 1.11: Coupled springs.

1.1.10.4 Ex: Coupled springs

A mass m is suspended by four springs with the constants k_n, as shown in the figure. Determine the equilibrium position of the mass. Assume the ideal case of ideally compressible springs.

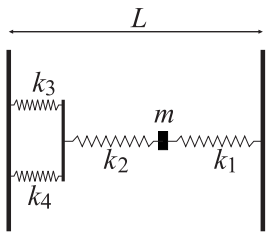

Figure 1.12: Coupled springs.

1.1.10.5 Ex: Coupled springs

Calculate the resulting spring constants for the constructions shown in the scheme. Individual springs are arbitrarily compressible with spring constants D_k.

1.1.10.6 Ex: Spring-mass system

A body of unknown mass hangs at the end of a spring, which is neither stretched nor compressed, and is released from rest at a certain moment. The body drops a distance y_1 until it rests for the first time after the release. Calculate the period of oscillatory motion.

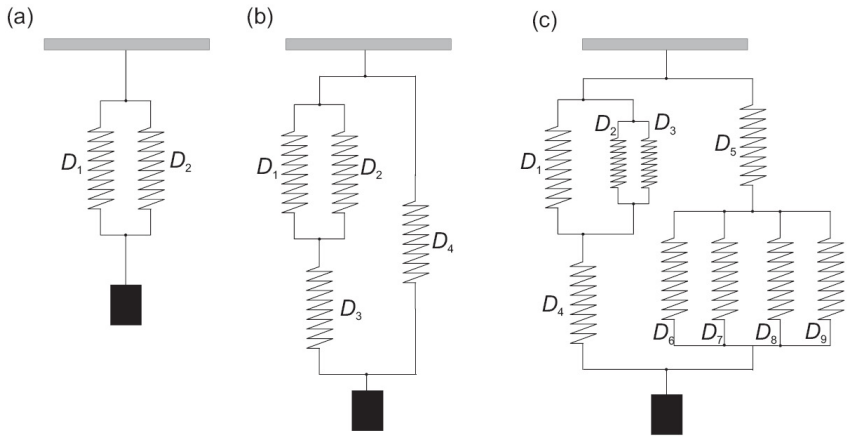

Figure 1.13: Coupled springs.

1.1.10.7 Ex: Spring-mass system

A body of $m = 1.5\,\text{kg}$ stretches a spring by $y_0 = 2.8\,\text{cm}$ from its natural length when being at rest. Now, we let it swing at this spring with an amplitude of $y_m = 2.2\,\text{cm}$.
a. Calculate total energy of the system.
b. Calculate the gravitational potential energy at the body's lower turning point.
c. Calculate the potential energy of the spring at the body's lower turning point.
d. What is the maximum kinetic energy of the body (when $U = 0$ is the point where the spring is at equilibrium).

1.1.10.8 Ex: U-shaped water tube

Consider a U-shaped tube filled with water. The total length of the water column is L. Exerting pressure on one tube outlet the column is incited to perform oscillations. Calculate the period of the oscillation.

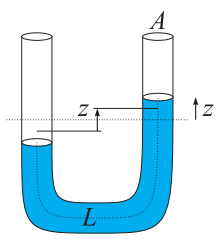

Figure 1.14: U-shaped water tube.

1.1.10.9 Ex: Buoy in the sea

A hollow cylindrical buoy with cross-sectional area A and mass M floats in the sea so that the axis of symmetry is aligned with gravitation. An albatross of mass m

sitting on the buoy waits until time $t = 0$ and takes off. With which frequency and amplitude does the buoy oscillate if friction can be neglected? Derive the equation of motion and the complete solution.

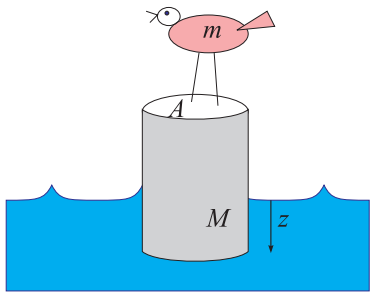

Figure 1.15: Fluctuating buoy.

1.1.10.10 Ex: Complicated pendulum oscillation

At a distance of $d = 30\,\text{cm}$ below the suspension point of a pendulum with the length $l_1 = 50\,\text{cm}$ there is a fixed pin S on which the wire suspending the pendulum temporarily bends during vibration. How many vibrations does the pendulum perform per minute?

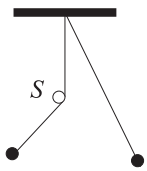

Figure 1.16: Mathematical pendulum.

1.1.10.11 Ex: Physical pendulum

Calculate the oscillation frequency of a thin bar of mass m and length L suspended at one end.

1.1.10.12 Ex: Physical pendulum

An irregularly shaped flat body has the mass $m = 3.2\,\text{kg}$ and is hung on a massless rod with adjustable length, which is free to swing in the plane of the body itself. When the rod's length is $L_1 = 1.0\,\text{m}$, the period of the pendulum is $t_1 = 2.6\,\text{s}$. When the rod is shortened to $L_2 = 0.8\,\text{m}$, the period decreases to $t_2 = 2.5\,\text{s}$. What is the period of the oscillation when the length is $L_3 = 0.5\,\text{m}$?

1.1.10.13 Ex: Physical pendulum

A physical pendulum of mass M consists of a homogeneous cube with the edge length d. As shown in the figure, the pendulum is hung without friction on a horizontal rotation axis.
a. Determine the inertial momentum about the rotation axis using Steiner's theorem.
b. The pendulum now performs small oscillations around its resting position. Determine the angular momentum.
c. Give the equation of motion for small pendulum amplitudes ϕ around its resting position and the oscillation period.

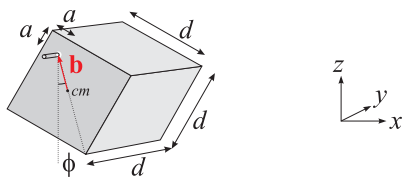

Figure 1.17: Physical pendulum.

1.1.10.14 Ex: Physical pendulum on a spiral spring

Consider a beam of mass $m = 1\,\mathrm{kg}$ with the dimensions $(a, b, c) = (3\,\mathrm{cm}, 3\,\mathrm{cm}, 8\,\mathrm{cm})$. The beam is rotatable about an axis through the point A. At point B, at a distance r from point A, the beam is fixed to a spiral spring exerting the retroactive force $\mathbf{F}_R = D\vec{\phi}$ with $D = 100\,\mathrm{N/m}$. Determine the differential equation of motion and solve it. Determine the period of the oscillation.

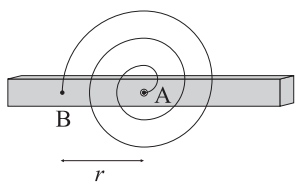

Figure 1.18: Physical pendulum on a spiral spring.

1.1.10.15 Ex: Accelerated pendulum

A simple pendulum of length L is attached to a cart that slides without friction downward an plane inclined by an angle α with respect to the horizontal. Determine the oscillation period of the pendulum on the cart.

1.1.10.16 Ex: Accelerated pendulum

a. A pendulum of length L and mass M is suspended from the roof of a wagon horizontally accelerated with the acceleration a_{ext}. Find the equilibrium position of the pendulum. Determine the oscillation frequency for small oscillations and derive

the differential equation of motion for an observer sitting in the wagon. (Note that you cannot assume small displacements, if the acceleration a_{ext} is large.)
b. In the same wagon there is a mass m connected to the front wall by a spring k. Find the equilibrium position of the mass. Determine the oscillation frequency and derive the differential motion equation for an observer sitting in the wagon.

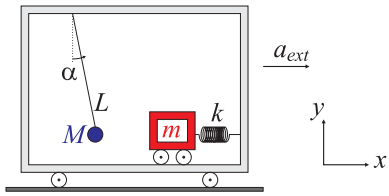

Figure 1.19: Accelerated pendulum.

1.1.10.17 Ex: Oscillation of a rolling cylinder

Consider a cylinder secured by two springs that rotates without sliding, as shown in the figure. Calculate the frequency for small oscillations of the system.

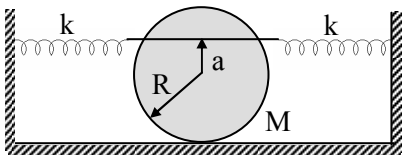

Figure 1.20: Rolling cylinder.

1.1.10.18 Ex: Rocking chair

Consider a thin rod of mass M and length $2L$ leaning on its center-of-mass, as shown in the figure. It is attached at both ends by springs of constants k. Calculate the angular frequency for small oscillations of the system.

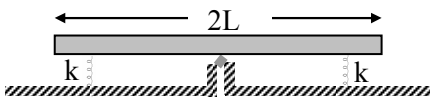

Figure 1.21: Rocking chair.

1.1.10.19 Ex: Rotational oscillation of a disk

Consider a disk of mass M and radius R ($I = \frac{1}{2}MR^2$) that can rotate around the polar axis. A body of mass $m$hangs at an ideal rope that runs through the disk (without slipping) and is attached to a wall by a spring of constant k, as shown in the figure. Calculate the natural oscillation frequency of the system.

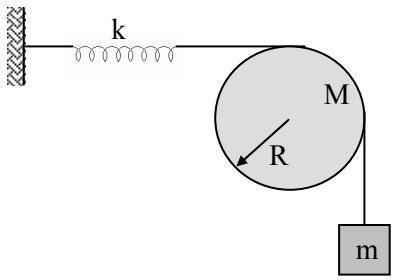

Figure 1.22: Rotational oscillation of a disk.

1.1.10.20 Ex: Oscillation of a half cylinder

Consider a massive, homogeneous half-cylinder of mass M and radius R resting on a horizontal surface. If one side of this solid is slightly pushed down and released, it will swing around its equilibrium position. Determine the period of this oscillation.

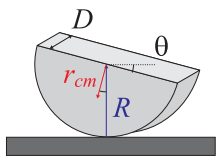

Figure 1.23: Oscillation of a half cylinder.

1.1.10.21 Ex: Pendulum coupled to a spring

Consider a simple pendulum of mass m and length L, connected to a spring of constant k, as shown in the figure. Calculate the frequency of the system for small oscillation amplitudes.

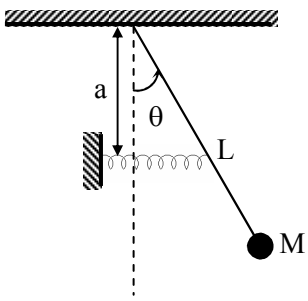

Figure 1.24: Pendulum coupled to a spring.

1.1.10.22 Ex: Pendulum carousel

A mass m is hung by a rope of length l on a carousel with the radius R. The pendulum performs small amplitude oscillations in the direction of the rotation axis of the carousel. How does the period of oscillation depend on the rotation speed of the carousel?

1.2 Superposition of periodic movements

Several movements that we already know can be understood as superpositions of periodic movements in different directions and, possibly, with different phases. Example are the circular or elliptical motion of a planet around the sun or the Lissajous figures. In these cases, the motion must be described by vectors, $\mathbf{r}(t) \equiv (x(t), y(t))$. It is also possible to imagine superpositions of periodic movements in the same degree of freedom. The movement of the membrane of a loudspeaker or musical instruments usually vibrates harmonically, but follows a *superposition* of harmonic oscillations. According to the *superposition principle*, we will take the resultant of several harmonic vibrations as the sum of the individual vibrations.

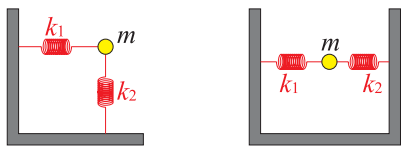

Figure 1.25: Superposition of vibrations in different (left) and equal (right) degrees of freedom.

1.2.1 Rotations and complex notation

We now consider a uniform circular motion. The radius of the circle being R, the motion is completely described by the angle $\theta(t)$ which grows uniformly,

$$\theta = \omega t + \alpha \ . \tag{1.51}$$

The projections of the movement in x and y are,

$$x(t) = A\cos\theta \qquad \text{and} \qquad y(t) = A\sin\theta \ . \tag{1.52}$$

Thus, we can affirm $x(t) = y(t + \pi/2)$, that is, the projections have a mutual phase shift of $\pi/2$.

The circular motion can be represented in the complex plane using the imaginary unit $i \equiv \sqrt{-1}$ and Euler's relationship $e^{\imath\theta} = \cos\theta + \imath\sin\theta$, as illustrated in Fig. 1.26.[3,4] With $r = Ae^{\imath\theta}$ we obtain $x = A\mathfrak{Re}\ e^{\imath\theta}$ and $\imath y = A\mathfrak{Im}\ e^{\imath\theta}$ and $r = x + \imath y$.

We will use the complex notation extensively, as it greatly facilitates the calculation.

[3]The Euler relation can easily be derived by Taylor expansion.

[4]To check your notions on complex numbers do the exercises in Chp. 1 of the Book of A.P. French.

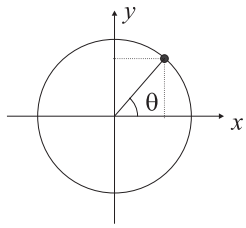

Figure 1.26: Circular motion in the complex plane.

1.2.2 Lissajous figures

Other periodic movements in the two-dimensional plane are possible, when the movements in x and y have different phases or frequencies. These are called *Lissajous figures*.

We consider a body subject to two harmonic movements in orthogonal directions:

$$x(t) = A_x \cos(\omega_x t + \varphi_x) \qquad \text{and} \qquad y(t) = A_y \cos(\omega_y t + \varphi_y) \ . \tag{1.53}$$

When ω_x/ω_y is a rational number, the curve is closed and the motion repeats after equal time periods. The upper charts in Fig. 1.27 show trajectories of the body for $\omega_x/\omega_y = 1/2$, $1/3$, and $2/3$, letting $A_x = A_y$ and $\varphi_x = \varphi_y$. The lower charts in Fig. 1.27 show trajectories for $\omega_x/\omega_y = 1/2$, $1/3$, letting $\varphi_x - \varphi_y = 0$, $\pi/4$, and $\pi/2$.

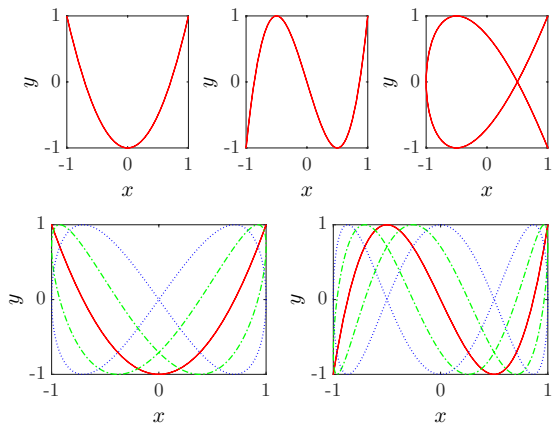

Figure 1.27: (code) Trajectories of a body oscillating with different frequencies in two dimensions.

Example 4 (*Lissajous figures*):

- Connect two function generators to the two channels of an oscilloscope in x-y.
- MATLAB simulation.

.

1.2.3 Vibrations with equal frequencies superposed in one dimension

Vibratory movements can overlap. The result can be described as a sum,

$$\begin{aligned} x(t) &= x_1(t) + x_2(t) = A_1 \cos(\omega t + \alpha_1) + A_2 \cos(\omega t + \alpha_2) \\ &= \mathrm{Re}[A_1 e^{\imath(\omega t + \alpha_1)} + A_2 e^{\imath(\omega t + \alpha_2)}] = \mathrm{Re} e^{\imath \omega t}[A_1 e^{\imath \alpha_1} + A_2 e^{\imath \alpha_2}] \ . \end{aligned} \tag{1.54}$$

That is, the new motion is a cosine vibration, $x(t) = A \cos \omega t$, with the phase,

$$\tan \alpha = \frac{\mathrm{Im}\ x(0)}{\mathrm{Re}\ x(0)} = \frac{\mathrm{Im}(A_1 e^{\imath \alpha_1} + A_2 e^{\imath \alpha_2})}{\mathrm{Re}(A_1 e^{\imath \alpha_1} + A_2 e^{\imath \alpha_2})} = \frac{A_1 \sin \alpha_1 + A_2 \sin \alpha_2}{A_1 \cos \alpha_1 + A_2 \cos \alpha_2} \ , \tag{1.55}$$

and the amplitude,

$$A = |A_1 e^{\imath \alpha_1} + A_2 e^{\imath \alpha_2}| = \sqrt{A_1^2 + A_2^2 + 2 A_1 A_2 \cos(\alpha_1 - \alpha_2)} \ . \tag{1.56}$$

We consider the case $A_1 = A_2$,

$$\tan \alpha = \frac{\sin \alpha_1 + \sin \alpha_2}{\cos \alpha_1 + \cos \alpha_2} \quad , \quad A = 2A \cos \frac{\alpha_1 - \alpha_2}{2} \ . \tag{1.57}$$

The cases $\alpha_1 = \alpha_2$ or $\alpha_2 = 0$ further simplify the result.

1.2.4 Frequency beat

Vibratory movements with different frequencies can overlap. The result can be described as a sum,

$$x(t) = x_1(t) + x_2(t) = A_1 \cos \omega_1 t + A_2 \cos \omega_2 t = \mathfrak{Re}\ [A_1 e^{\imath \omega_1 t} + A_2 e^{\imath \omega_2 t}] \ . \tag{1.58}$$

Considering the case $A_1 = A_2$ we obtain,

$$\begin{aligned} x(t) &= A \mathfrak{Re}\ [e^{\imath \omega_1 t} + e^{\imath \omega_2 t}] \\ &= A \mathfrak{Re}\ [e^{\imath(\omega_1 + \omega_2)t/2} e^{\imath(\omega_1 - \omega_2)t/2} + e^{\imath(\omega_1 + \omega_2)t/2} e^{-\imath(\omega_1 - \omega_2)t/2}] \\ &= A \mathfrak{Re}\ e^{\imath(\omega_1 + \omega_2)t/2} 2 \cos \frac{(\omega_1 - \omega_2)t}{2} \\ &= 2A \cos \frac{(\omega_1 + \omega_2)t}{2} \cos \frac{(\omega_1 - \omega_2)t}{2} \ . \end{aligned} \tag{1.59}$$

Example 5 (*Amplitude modulation*): An important example is the amplitude modulation of radiofrequency signals.

Example 6 (*Visualization of beat frequencies on an oscilloscope*):

- Connect two function generators to the two channels of an oscilloscope and add the channels.
- MATLAB simulation.
- Modulate one signal by another in a frequency mixer.

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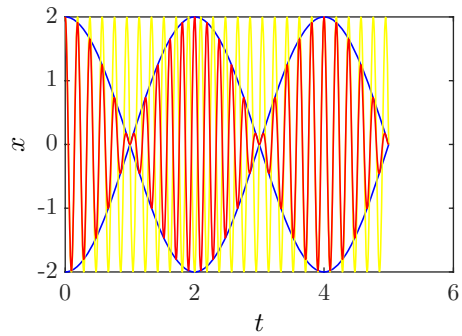

Figure 1.28: (code) Illustration of the beating of two frequencies $\nu_1 = 5.5\,\mathrm{Hz}$ and $\nu_2 = 5\,\mathrm{Hz}$ showing the perceived vibration (red), the vibration with the frequency $(\nu_1 + \nu_2)/2$ (blue), and the vibration with frequency $(\nu_1 - \nu_2)/2$ (yellow).

1.2.5 Amplitude and frequency modulation

Radio frequencies above 300 kHz can easily be emitted and received by antennas, while audio frequencies are below 20 kHz. However, radio frequencies can be used as carriers for audio frequencies. This can be done by modulating the audio signal on the *amplitude* of the carrier () before sending the carrier frequency. The receiver retrieves the audio signal by demodulating the carrier. Therefore, audio signals can be transmitted by electromagnetic waves. Another technique consists in modulating the *frequency* of these waves (). We will now calculate the spectrum of these two modulations using complex notation and show how to demodulate the encoded audio signals by multiplication with a local oscillator corresponding to the carrier wave.

1.2.5.1 AM

Let ω and Ω be the frequencies of the carrier wave and the modulation, respectively. We can describe the amplitude modulation by,

$$U(t) = (1 + S(t)) \cos \omega t \ . \tag{1.60}$$

After the receiver has registered this signal, we demodulate it by multiplying it with $\cos \omega t$:

$$U(t) \cos \omega t = (1 + S(t)) \cos^2 \omega t = (1 + S(t)) \left(\tfrac{1}{2} + \tfrac{1}{2} \cos 2\omega t \right) \ . \tag{1.61}$$

We purify this signal passing it through a low-pass filter eliminating the rapid oscillations:

$$U(t) \cos \omega t \longrightarrow \tfrac{1}{2}(1 + S(t)) \ . \tag{1.62}$$

We retrieve the original signal $S(t)$.

1.2.5.2 FM

We can describe the frequency modulation by,

$$U(t) = e^{\imath(\omega t + N \sin \Omega t)} = e^{\imath \omega t} \sum_{k=-\infty}^{\infty} \mathcal{J}_k(N) e^{\imath k \Omega t} \ . \tag{1.63}$$

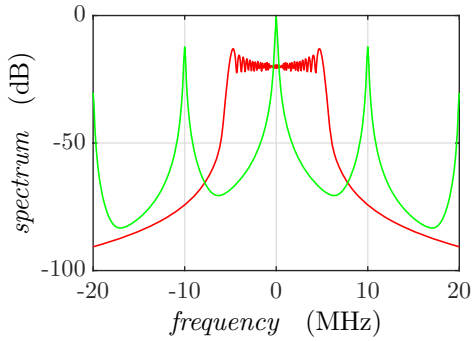

Figure 1.29: (code) Modulation signal.

The modulation of the carrier wave generates sidebands. This can be seen by expanding the signal carrying the phase modulation into a Fourier series,

$$e^{\imath\omega t}\sum_{k=-\infty}^{\infty}\mathcal{J}_k(\beta)e^{\imath k\Omega t}\simeq e^{\imath\omega t}+\mathcal{J}_1(N)e^{\imath\omega t+\imath\Omega t}+\mathcal{J}_{-1}(N)e^{\imath\omega t-\imath\Omega t} \tag{1.64}$$

when the *modulation index* N is small. Here, $J_{-k}(N)=(-1)^kJ_k(N)$ are the Bessel functions.

The spectrum of a signal with PM modulation consists of discrete lines, called sidebands, whose amplitudes are given by Bessel functions,

$$S(\omega)=\sum_{k=-\infty}^{\infty}|\mathcal{J}_k(N)|^2\delta(\omega+k\Omega)\ . \tag{1.65}$$

In real systems, the frequency bands have finite widths β due to frequency noise or to the finite resolution of the detectors,

$$S(\omega)=\sum_{k=-\infty}^{\infty}|\mathcal{J}_k(N)|^2\frac{N^2}{(\omega-k\Omega)^2+N^2}\ . \tag{1.66}$$

Example 7 (*Frequency spectrum*):

- Modulate the frequency of a VCO.
- Show in the spectrum analyzer the transition to sidebands.

.

1.2.6 Exercises

1.2.6.1 Ex: Amplitude modulation

Consider a carrier wave of $\omega/2\pi=1$ MHz frequency whose amplitude is modulated by an acoustic signal of $\Omega/2\pi=1\,\mathrm{kHz}$: $U(t)=A\cos\Omega t\cos\omega t$. To demodulate the signal, multiply the received wave $U(t)$ by the carrier radiofrequency. Interpret the result.

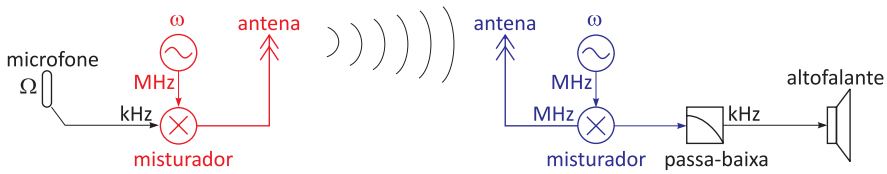

Figure 1.30: Illustration of radiofrequency signal transmission.

1.3 Damped and forced vibrations

Frequently, vibrations are exposed to external perturbations. For example, damping forces due to friction exerted by the medium in which vibration takes place work to waste and dissipate the energy of the oscillation and, therefore, to reduce the amplitude of the oscillation. In contrast, periodic forces can pump energy into the oscillator system and excite vibrations.

1.3.1 Damped vibration and friction

Let us first deal with damping by forces named *Stokes friction*, that is, forces which are proportional to the velocity of the oscillating mass and contrary to the direction of motion, $F_{frc} = -bv$, where b is the friction coefficient. With this additional term, the equation of motion is,

$$ma = -bv - kx \ . \tag{1.67}$$

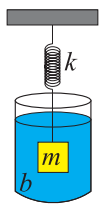

Figure 1.31: Oscillation damped by a viscous medium.

The calculation of the *damped oscillator* can be greatly simplified by the use of complex numbers by making the ansatz,

$$x(t) = Ae^{\lambda t} \ , \tag{1.68}$$

where λ is a complex number. We get,

$$m\lambda^2 + b\lambda + k = 0 \ , \tag{1.69}$$

giving the characteristic equation,

$$\lambda = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \ , \tag{1.70}$$

with

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \text{and} \qquad \gamma = \frac{b}{2m} \ . \tag{1.71}$$

The friction determines the damping behavior. We distinguish three cases discussed in the following sections.

1.3.1.1 Overdamped case

In the overdamped case, for $\omega_0 < \gamma$, there are two real solutions $\lambda = -\gamma \pm \kappa$ with $\kappa \equiv \sqrt{\gamma^2 - \omega_0^2}$ for the characteristic equation, giving,

$$x(t) = e^{-\gamma t}(Ae^{-\kappa t} + Be^{\kappa t}) \ . \tag{1.72}$$

Choosing the initial conditions,

$$\begin{aligned} x_0 = x(0) &= e^{-\gamma t}(Ae^{-\kappa t} + Be^{\kappa t}) = A + B \\ 0 = v(0) &= -A(\gamma+\kappa)e^{-(\gamma+\kappa)t} - B(\gamma-\kappa)e^{-(\gamma-\kappa)t} = -A(\gamma+\kappa) - B(\gamma-\kappa) \ , \end{aligned} \tag{1.73}$$

we determine the amplitudes,

$$A = \frac{x_0}{2}\left(1 - \frac{\gamma}{\kappa}\right) \qquad \text{and} \qquad B = \frac{x_0}{2}\left(1 + \frac{\gamma}{\kappa}\right) \ . \tag{1.74}$$

Finally, the solution is [5],

$$x(t) = x_0 e^{-\gamma t}\left[\cosh \kappa t + \frac{\gamma}{\kappa}\sinh \kappa t\right] \ . \tag{1.75}$$

1.3.1.2 Underdamped case

In the underdamped case, for $\omega_0 > \gamma$, we have two complex solutions $\lambda = -\gamma \pm \imath\omega$ with $\omega \equiv \sqrt{\omega_0^2 - \gamma^2}$, giving,

$$x(t) = e^{-\gamma t}(Ae^{\imath\omega t} + Be^{-\imath\omega t}) \ . \tag{1.76}$$

Choosing the initial conditions,

$$\begin{aligned} x_0 = x(0) &= e^{-\gamma t}(Ae^{\imath\omega t} + Be^{-\imath\omega t}) = A + B \\ 0 = v(0) &= -A(\gamma-\imath\omega)e^{-(\gamma-\imath\omega)t} - B(\gamma+\imath\omega)e^{-(\gamma+\imath\omega)t} = -A(\gamma-\imath\omega) - B(\gamma+\imath\omega) \ , \end{aligned} \tag{1.77}$$

we determine the amplitudes,

$$A = \frac{x_0}{2}\left(1 + \frac{\gamma}{\imath\omega}\right) \qquad \text{and} \qquad B = \frac{x_0}{2}\left(1 - \frac{\gamma}{\imath\omega}\right) \ . \tag{1.78}$$

Finally, the solution is [6],

$$x(t) = x_0 e^{-\gamma t}\left[\cos \omega t + \frac{\gamma}{\omega}\sin \omega t\right] \ . \tag{1.79}$$

[5] Note that for super-strong damping, we have $\kappa \simeq \gamma$ and therefore,

$$x(t) = Ae^{-2\gamma t} + B \ .$$

, This is nothing more than the solution of the equation of motion without restoring force, $ma = -bv$.

[6] Note that, for very weak damping, we have $\gamma \simeq 0$ and $\omega \simeq \omega_0$ and hence,

$$x(t) = Ae^{\imath\omega_0 t} + Be^{-\imath\omega_0 t} \ .$$

This is nothing more than the solution of the frictionless equation of motion, $ma = -kx$.

1.3.1.3 Critically damped case

In the critically damped case, for $\omega_0 = \gamma$, there is only one solution $\lambda = -\gamma$, giving

$$x(t) = Ae^{-\gamma t} \ . \tag{1.80}$$

Since one solution is not sufficient to solve a second order differential equation, we need to look for another linearly independent solution. We can try another ansatz,

$$x(t) = Bte^{\lambda t} \ , \tag{1.81}$$

resulting in the characteristic equation,

$$m(\lambda^2 te^{\lambda t} + 2\lambda e^{\lambda t}) + b(\lambda te^{\lambda t} + e^{\lambda t}) + kte^{\lambda t} = 0 \ . \tag{1.82}$$

The terms in $e^{\lambda t}$ and $te^{\lambda t}$ should disappear separately, giving,

$$2m\lambda + b = 0 \quad \text{and} \quad m\lambda^2 t + b\lambda t + kt = 0 \quad \Longrightarrow \quad \lambda = -\frac{b}{2m} = -\gamma = -\omega_0 \ . \tag{1.83}$$

Finally, the solution is,

$$x(t) = (A + Bt)e^{-\gamma t} \ . \tag{1.84}$$

Choosing the initial conditions,

$$\begin{aligned} x_0 = x(0) &= (A + Bt)e^{-\gamma t} = A \\ 0 = v(0) &= (-\gamma A - \gamma Bt + B)e^{-\gamma t} = -\gamma A + B \ , \end{aligned} \tag{1.85}$$

we determine the amplitudes,

$$A = x_0 \quad \text{and} \quad B = \gamma x_0 \ . \tag{1.86}$$

Finally, the solution is,

$$x(t) = x_0(1 + \gamma t)e^{-\gamma t} \ . \tag{1.87}$$

Fig. 1.32 illustrates the damping of the oscillation for various friction rates γ.

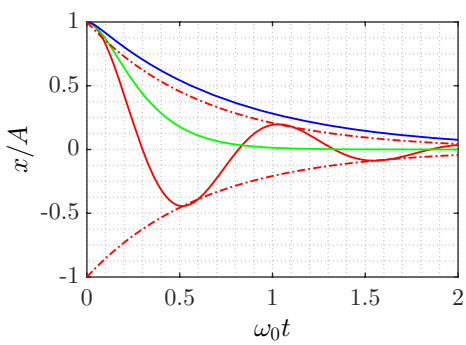

Figure 1.32: (code) Damped oscillation for $\omega_0 = 10\,\mathrm{s}^{-1}$ and $\gamma = 2.5\,\mathrm{s}^{-1}$ (red), $10\,\mathrm{s}^{-1}$ (green), and $25\,\mathrm{s}^{-1}$ (blue).

The critical friction coefficient generates a damped movement without any 'overshoot', since the velocity $\dot{x}(t)$ only disappears for $t = 0$.

1.3.1.4 Quality factor and energy loss

For a harmonic oscillation we establish the balance of energies,

$$\begin{aligned} E &= \frac{m}{2}v^2 + \frac{k}{2}x^2 \\ &= \frac{m}{2}\left(A\imath\omega_0 e^{\imath\omega_0 t} - B\imath\omega_0 e^{-\imath\omega_0 t}\right)^2 + \frac{m}{2}\omega_0^2\left(Ae^{\imath\omega_0 t} + Be^{-\imath\omega_0 t}\right)^2 = 2m\omega_0^2 AB \ . \end{aligned} \tag{1.88}$$

Now, for an underdamped oscillation we replace the amplitudes by $A \to Ae^{-\gamma t}$ and $B \to Be^{-\gamma t}$, such that,

$$E(t) = 2m\omega_0^2 ABe^{-2\gamma t} \ . \tag{1.89}$$

Obviously, the energy is decreasing at the rate 2γ.

We define the *quality factor* as the number of radians that the damped system oscillates before its energy falls to e^{-1},

$$Q = \frac{\omega}{2\gamma} = \frac{\omega m}{b} \simeq \frac{\omega_0 m}{b} \ . \tag{1.90}$$

Comparing the initial energy with the energy remaining after one cycle,

$$\frac{E}{\Delta E} = \frac{E(0)}{E(0) - E(2\pi/\omega)} = \frac{1}{1 - e^{-4\pi\gamma/\omega}} \simeq \frac{\omega}{4\pi\gamma} \ , \tag{1.91}$$

we find that the quantity,

$$\frac{Q}{2\pi} = \frac{E}{\Delta E} \tag{1.92}$$

represents a measure for the energy dissipation.

1.3.2 Forced vibration and resonance

We have seen that a damped oscillator loses its energy over time. To sustain the oscillation, it is necessary to provide energy. The simplest way to do this, is to force the oscillator to oscillate at a frequency ω by applying an external force $F_0 \cos\omega t$. The question now is, what will be the amplitude of the oscillation and its phase with respect to the phase of the applied force. We begin by establishing the equation of motion,

$$ma + bv + m\omega_0^2 x = F_0 \cos\omega t \ . \tag{1.93}$$

The calculation can be greatly simplified by the use of complex numbers. We write the differential equation as,

$$ma + bv + m\omega_0^2 x = F_0 e^{\imath\omega t} \ , \tag{1.94}$$

making the ansatz $x(t) = Ae^{\imath\omega t - \imath\delta}$, yielding

$$-\omega^2 Ae^{\imath\omega t - \imath\delta} m + \imath\omega b Ae^{\imath\omega t - \imath\delta} + m\omega_0^2 Ae^{\imath\omega t - \imath\delta} = F_0 e^{\imath\omega t} \ . \tag{1.95}$$

We rewrite this formula,

$$e^{\imath\delta} = A\frac{m(\omega_0^2 - \omega^2) + \imath b\omega}{F_0} \ . \tag{1.96}$$

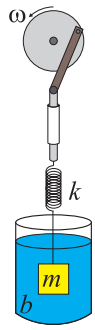

Figure 1.33: Forced oscillation damped by a viscous medium.

Immediately we get the solutions,

$$\tan\delta = \frac{\sin\delta}{\cos\delta} = \frac{\mathfrak{Im}\ e^{\imath\delta}}{\mathrm{Re}e^{\imath\delta}} = \frac{b\omega}{m(\omega_0^2-\omega^2)} \tag{1.97}$$

$$A = \left|Ae^{-\imath\delta}\right| = \left|\frac{F_0}{m(\omega_0^2-\omega^2)+\imath\omega b}\right| = \frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2+b^2\omega^2}}\ .$$

The frequency response (spectrum) of the oscillator to the periodic excitation is illustrated in Fig. 1.34. We see that, when we increase the friction, we decrease the height and increase the width of the spectrum $|A(\omega)|$. Fig. 1.34(b) shows that, increasing the excitation frequency, the oscillation undergoes a phase shift of π.

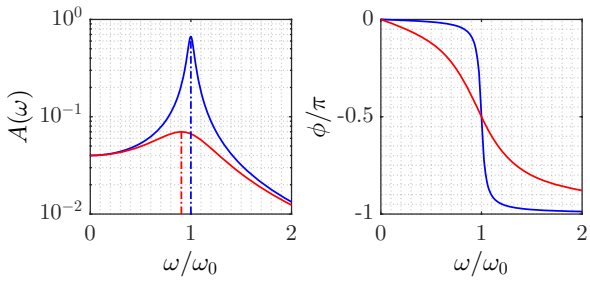

Figure 1.34: (code) Frequency response of the amplitude and phase of the oscillator for a force of $F_0 = 1\,\mathrm{N}$, a mass of $m = 1\,\mathrm{kg}$, a resonance frequency of $\omega_0 = (2\pi)\ 5\,\mathrm{Hz}$, and a friction coefficient of $b = 0.5$ (blue) or $b = 1$ (red).

We now ask, at what excitation frequency ω the oscillator responds with maximum amplitude,

$$0 = \frac{d}{d\omega_m}A(\omega_m) = F_0\omega_m\frac{2m^2\omega_0^2-2m^2\omega_m^2-b^2}{(m^2\omega_0^4-2m^2\omega_0^2\omega_m^2+m^2\omega_m^4+b^2\omega_m^2)^{\frac{3}{2}}}\ . \tag{1.98}$$

The numerator disappears for,

$$\omega_m = \sqrt{\omega_0^2-\frac{b^2}{2m^2}}\ , \tag{1.99}$$

and the amplitude becomes,

$$A_m = \frac{F_0}{b\sqrt{\omega_0^2 - \frac{b^2}{4m^2}}} \ . \tag{1.100}$$

1.3.2.1 Quality factor

For weak damping $\gamma \ll \omega_0$ and small detunings, $|\omega - \omega_0| \ll \omega_0$, we can approximate the expression for the spectrum by,

$$A(\omega) \simeq \left| \frac{F_0}{m} \frac{1}{2\omega_0(\omega_0 - \omega) + \imath\omega_0 \frac{b}{m}} \right| = \left| \frac{F_0}{2m\omega_0} \frac{1}{\omega - \omega_0 - \imath\gamma} \right| \ .$$

This function corresponds to a Lorentzian profile with the width FWHM $\Delta\omega = 2\gamma$. The *quality factor* defined in the section discussing the damped oscillator measures the quality of the resonance,

$$Q = \frac{\omega}{2\gamma} = \frac{\omega}{\Delta\omega} \ . \tag{1.101}$$

Example 8 (*Harmonic vibration*):

- Construct a L-C-circuit, excite it by a function generator by making a frequency ramp, and show the resonance on the oscilloscope. It works with a coil of $N = 12$ turns, of length $\ell = 6\,\mathrm{cm}$ and of radius $r = 1.4\,\mathrm{cm}$, giving $L = 1.4\,\mu\mathrm{H}$. We can also set $R = 2.2\,\Omega$ and $C = 100\,\mathrm{nF}$, giving $\omega_0 = 9.4\,\mathrm{MHz}$.

1.3.3 Exercises

1.3.3.1 Ex: Resolution of the damped oscillator equation

Solve the damped oscillator equation for $4km > b^2$ using the ansatz $x(t) = Ae^{-\gamma t}\cos\omega t$.

1.3.3.2 Ex: Damped oscillation

In a damped oscillation the oscillation period is $T = 1\,\mathrm{s}$. The ratio between two consecutive amplitudes is 2. Despite the large damping, the deviation of the period T_0 compared to the undamped oscillation is small. Calculate the deviation.

1.3.3.3 Ex: Damped physical pendulum

The physical pendulum shown in the figure consists of a disk of mass M and radius R suspended on an axes parallel to the symmetry axis of the disk and passing the edge of the disk.
a. Calculate the inertial momentum of the disk, $I = \int_V r^2 dm$, with respect to the suspension axes.
b. Derive the equation of motion by considering a weak Stokes damping due to friction proportional to the angular velocity and by approximating for small amplitude oscillations.

c. What is the natural oscillation frequency of the pendulum (without friction)? How to calculate the oscillation frequency considering friction?
d. Write down the solution of the equation of motion for the initial situation $\phi(0) = 0$ and $\dot{\phi}(0) = \dot{\phi}_0$.

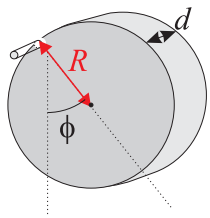

Figure 1.35: Damped physical pendulum.

1.3.3.4 Ex: Pendulum with friction

Jane has prepared dinner and Tarzan (80 kg) and Cheeta (40 kg) must return home. The house is in a tree at a height of 10 m, so that both must swing home on a (massless) rope hanging from $l = 100$ m high tree. Tarzan grabs the rope at the height of its center-of-mass $h = 1.2$ m above ground, Cheeta because of its height is smaller at 0.8 m above ground. With what initial speed both need to grab the rope to reach the platform of the house with their feet. Consider Stokes' friction force, $F_R = C \cdot v$ with $C = 4 \cdot 10^{-4}$ Ns/m (Tarzan) respectively, $C = 2 \cdot 10^{-4}$ Ns/m (Cheeta). Why is this force different for the two? Treat the oscillating motion as small displacement. Determine whether the vibration is weakly damped. Do you think Jane will have dinner alone?

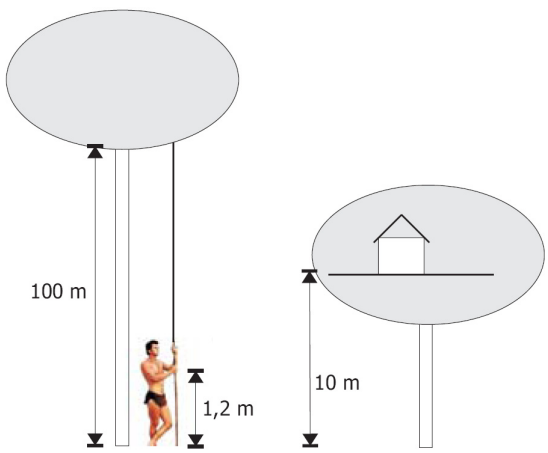

Figure 1.36: Pendulum with friction.

1.3.3.5 Ex: Resolution of the forced oscillator equation

Solve the forced oscillator equation using the ansatz $x(t) = A\cos(\omega t - \delta)$.

1.3.3.6 Ex: Oscillation with coercive force

On a body of mass m along the x-axis act a force proportional to the displacement $F_h = -\kappa x$ and a Stokes friction force $F_R = -\gamma\dot{x}$. A time-dependent force is switched on at time $t = 0$, while the body rests at the position $x = 0$. The force increases linearly over time until it suddenly disappears at time $t = T$. Determine the work that the external force has done up this time. Consider the various solutions of the equation of motion resulting from the various combinations of κ and γ.

1.3.3.7 Ex: Oscillation with coercive force

You want to measure the friction coefficient γ of a sphere (mass $m = 10\,\mathrm{kg}$, diameter $d = 10\,\mathrm{cm}$) in water. To do this, you let the sphere oscillate on a spring (spring constant $k = 100\,\mathrm{N/m}$) in a water bath exciting the oscillation by a periodic force, $F(t) = F_0 \cos\omega t$. By varying the excitation frequency ω until observing the maximum oscillation amplitude, you measure the resonance frequency $\omega_w = 2\pi \cdot 1\,\mathrm{Hz}$. Now, you let the water out of the tub and repeat the measurement finding $\omega_0 = 2\pi \cdot 2\,\mathrm{Hz}$.
a. Determine the resting position of the mass in water and air.
b. Establish the differential equation of motion. Assume that the weight of the sphere in water is reduced by the buoyancy $V\rho_{wat}g$, where V is the volume of the sphere and ρ_{wat} the density of the water.
c. What is the value of γ?

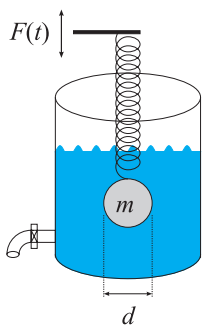

Figure 1.37: Driven pendulum.

1.3.3.8 Ex: Electronic oscillator circuit

The instantaneous current $I(t)$ in an L-R-C-circuit (inductance of a coil, ohmic resistance and capacitance in series) excited by an alternating voltage source $U(t) = U_0 \cos\omega t$ satisfies the following differential equation,

$$L\dot{I} + RI + C^{-1}\int_0^t I dt' = U_0 \sin\omega t \ .$$

a. Derive the equation for the moving charge $\dot{Q} = I$, compare the obtained equation with that of the damped and forced spring-mass oscillator and determine the solution for the current.
b. Determine the resonance frequency ω_0 of the circuit.
c. Determine the quality factor Q of the circuit. How you can increase Q without changing the resonance frequency?

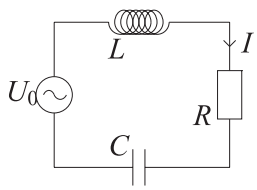

Figure 1.38: Line filter.

1.3.3.9 Ex: Electronic oscillator circuit

A voltage $U(t)$ is known to produce in an coil of inductance L the current $I_L = L^{-1}\int_0^{t'} U dt$, in an ohmic resistance R the current $I_R = R^{-1}U$, and in a capacitor of capacitance C the current $I = C\dot{U}$. In the parallel L-R-C circuit shown in the figure, at each instant of time the sum of the currents I_L, I_R and I_C must compensate the current $I_F(t) = I_0 e^{i\omega t}$ supplied by an alternating current source, while the voltage $U(t)$ is the same across all components.
a. Derive the differential equation for the derivative of the voltage $\dot{U}$.
b. What would be the oscillation frequency of the current without source ($I_0 = 0$) and without resistance ($R = \infty$)?
c. What would be the oscillation frequency of the current without source ($I_0 = 0$) but with resistance ($R \neq \infty$)?
d. Doing the ansatz $U(t) = U_0 e^{\imath\omega t + \imath\phi}$ derive the characteristic equation.
e. Use the characteristic equation to calculate the impedance defined by $Z \equiv |U_0/I_0|$ and the phase ϕ of the current oscillation as a function of the frequency ω. Prepare qualitative sketches of functions $Z(\omega)$ and $\phi(\omega)$.

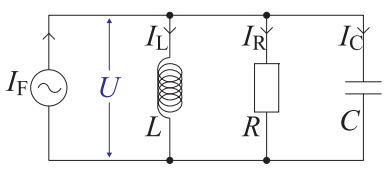

Figure 1.39: Notch filter.

1.3.3.10 Ex: Lorentz model of light-atom interaction

The Lorentz model describes the interaction of an electron attached to an atom with an incident light beam as a damped oscillator. The electron's binding to the nucleus

is taken into account by a restoring force $-\omega_0^2 x$. The decay of the excited state with the rate Γ is the reason for the damping force $-m\Gamma\dot{x}$. And the excitation is produced by the Lorentz force exerted by the electrical component of the light beam, $e\mathcal{E}_0 e^{\imath\omega t}$, where e is the charge of the electron. Establish the differential equation and calculate the amplitude of electron's oscillation as a function of the excitation frequency.

1.3.3.11 Ex: Lorentz model of light-atom interaction

a. Electric fields $\mathcal{E}$ exert on electric charges q the Coulomb force $F = q\mathcal{E}$. Write the differential equation for the undamped motion of an electron (charge $-e$, mass m) harmonically bound to its nucleus under the influence of an alternating electric field, $\mathcal{E} = \mathcal{E}_0 \sin\omega t$.
b. Show that the general solution can be written as,

$$x(t) = \frac{-e\mathcal{E}_0 \sin\omega t}{m(\omega_0^2 - \omega^2)} + A\cos\omega_0 t + B\sin\omega_0 t \ .$$

c. Write the solution in terms of the initial conditions $x(0) = 0 = \dot{x}(0)$.

1.4 Coupled oscillations and normal modes

So far we have discussed the behavior of isolated oscillators. Energy losses or gains were described in a bulk way via a coupling to an external reservoir without structure of its own. However, the reservoir often has vibrational degrees of freedom, as well, and can dump (or supply) energy. This usually happens when neighboring oscillators share a rigid, massive, or sturdy medium. The transfer of energy to neighboring oscillators is the key ingredient for any oscillatory propagation of energy called *wave*.

1.4.1 Two coupled oscillators

To discuss the *coupling between oscillators* at the most fundamental level, we consider two ideal and identical pendulums (length L and mass m) coupled by a spring of constant k, as shown in Fig. 1.40. The differential equations of motion for the angles

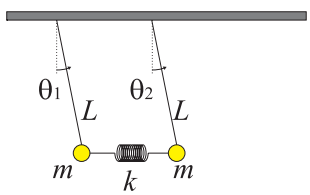

Figure 1.40: Two coupled pendulums.

θ_1 and θ_2 are,

$$\begin{aligned} mL\ddot{\theta}_1 &= -mg\sin\theta_1 - k(x_1 - x_2) \\ mL\ddot{\theta}_2 &= -mg\sin\theta_2 - k(x_2 - x_1) \ , \end{aligned} \tag{1.102}$$

with $x_j = L\sin\theta_j$. For small oscillations we have, therefore,

$$\begin{aligned}\ddot{\theta}_1 &= -\tfrac{g}{L}\sin\theta_1 - \tfrac{k}{m}(\sin\theta_1 - \sin\theta_2) \simeq -(\tfrac{g}{L} + \tfrac{k}{m})\theta_1 + \tfrac{k}{m}\theta_2 \\ \ddot{\theta}_2 &= -\tfrac{g}{L}\sin\theta_2 - \tfrac{k}{m}(\sin\theta_2 - \sin\theta_1) \simeq -(\tfrac{g}{L} + \tfrac{k}{m})\theta_2 + \tfrac{k}{m}\theta_1 \ .\end{aligned} \tag{1.103}$$

We define the normal coordinates of the vibration $\aleph \equiv \frac{1}{\sqrt{2}}(\theta_1 - \theta_2)$ and $\Psi \equiv \frac{1}{\sqrt{2}}(\theta_1 + \theta_2)$. We find the differential equations for $\aleph$ and Ψ by adding and subtracting the equations of motion,

$$\ddot{\theta}_1 + \ddot{\theta}_2 \simeq -\tfrac{g}{L}(\theta_1 + \theta_2) \qquad \text{and} \qquad \ddot{\theta}_1 - \ddot{\theta}_2 \simeq -(\tfrac{g}{L} + \tfrac{2k}{m})(\theta_1 - \theta_2) \ ,$$

or,

$$\ddot{\Psi} + \omega_\Psi^2 \Psi = 0 \qquad \text{and} \qquad \ddot{\aleph} + \omega_\aleph^2 \aleph = 0$$

using the angular frequencies of the vibrational normal modes,

$$\omega_\Psi = \sqrt{\frac{g}{L}} \qquad \text{and} \qquad \omega_\aleph = \sqrt{\frac{g}{L} + \frac{2k}{m}} \ .$$

1.4.2 Normal modes

Thus, the normal coordinates Ψ and $\aleph$ allow a description of the motion by decoupled linear differential equations. A vibration involving only one *normal coordinate* is called *normal mode*. In this mode all the components participating in the oscillation oscillate at the same frequency.

The importance of the normal modes is they are totally independent, that is, they never exchange energy and they can be pumped separately. Therefore, the total energy of the system can be expressed as the sum of terms containing the squares of the normal coordinates (potential energy) and their first derivatives (kinetic energy). Every independent path by which a system can gain energy is called *degree of freedom* and has an associated normal coordinate. For example, an isolated harmonic oscillator has two degrees of freedom, as it can gain potential or kinetic energy and two normal coordinates, x and v. And the coupled oscillator system,

$$E_\aleph = a\dot{\aleph}^2 + b\aleph^2 \qquad \text{and} \qquad E_\Psi = a\dot{\Psi}^2 + b\Psi^2 \ , \tag{1.104}$$

has four degrees of freedom.[7]

Every movement of the system can be represented by a superposition of normal modes,

$$\aleph = \tfrac{1}{\sqrt{2}}(\theta_1 - \theta_2) = \aleph_0 \cos(\omega_\aleph t + \phi_\aleph) \qquad \text{and} \qquad \Psi = \tfrac{1}{\sqrt{2}}(\theta_1 + \theta_2) = \Psi_0 \cos(\omega_\Psi t + \phi_\Psi) \ . \tag{1.105}$$

Choosing $\sqrt{2}A = \aleph_0 = \Psi_0$ and $\phi_\aleph = \phi_\Psi = 0$,

$$\begin{aligned}\theta_1 &= \tfrac{1}{\sqrt{2}}(\Psi + \aleph) = A\cos\omega_\aleph t + A\cos\omega_\Psi t = 2A\cos\tfrac{(\omega_\Psi - \omega_\aleph)t}{2}\cos\tfrac{(\omega_\Psi + \omega_\aleph)t}{2} \\ \theta_2 &= \tfrac{1}{\sqrt{2}}(\Psi - \aleph) = A\cos\omega_\aleph t - A\cos\omega_\Psi t = 2A\sin\tfrac{(\omega_\Psi - \omega_\aleph)t}{2}\sin\tfrac{(\omega_\Psi + \omega_\aleph)t}{2} \ .\end{aligned} \tag{1.106}$$

[7] Note that the motion of a single pendulum is a movement in two Cartesian dimensions and therefore would have four degrees of freedom. However, the joint action of gravity and the tension of the wire constrains the movement into one dimension thus freezing two degrees of freedom.

The oscillation shows the behavior of a frequency beat [8].

Example 9 (*Normal modes*):

- Two pendulums suspended on a movable horizontal bar which, in turn, is suspended by two wires to a rigid ceiling. Show (anti-)symmetric modes and their different exposure to damping of the motion of the bar.

1.4.3 Normal modes in large systems

There are techniques for solving systems many coupled oscillator. Let us consider, for example, a chain of $n = 1, ..., N$ oscillators coupled by springs. We have,

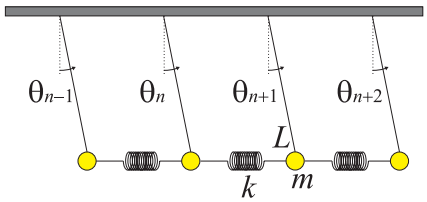

Figure 1.41: Array of coupled pendulums.

$$\ddot{\theta}_n = -\frac{g}{l}\theta_n - \frac{k}{m}(\theta_n - \theta_{n+1}) - \frac{k}{m}(\theta_n - \theta_{n-1}) \ . \tag{1.107}$$

Inserting the ansatz $\theta_n \equiv A_n e^{i\omega t}$, we obtain

$$\omega^2 A_n = \omega_0^2 A_n + \beta^2(A_n - A_{n+1}) + \beta^2(A_n - A_{n-1}) \ , \tag{1.108}$$

using the abbreviations $\omega_0^2 = g/l$ and $\beta^2 = k/m$. Defining the vector $\vec{A} \equiv (\cdots A_n \cdots)$ and the matrix,

$$\hat{M} \equiv \begin{pmatrix} \omega_0^2 + \beta^2 & -\beta^2 & & & & \\ -\beta^2 & \ddots & \ddots & & & \\ & \ddots & \omega_0^2 + 2\beta^2 & -\beta^2 & & \\ & & -\beta^2 & \omega_0^2 + 2\beta^2 & \ddots & \\ & & & \ddots & \ddots & -\beta^2 \\ & & & & -\beta^2 & \omega_0^2 + \beta^2 \end{pmatrix} \ , \tag{1.109}$$

we put the characteristic equation into a form called an eigenvalue equation,

$$\hat{M}\vec{A} = \omega^2 \vec{A} \ . \tag{1.110}$$

The matrix $\hat{M}$ is characterized by the fact that it contains on its diagonal the energy of each individual oscillator (that is, $\omega_0^2 + 2\beta^2$ when the oscillator is in the middle of

[8] Normal modes are observed in the molecular vibrations of H_2O and CO_2 (see Pain).

the chain, and $\omega_0^2 + \beta^2$) at the two ends of the chain). On the secondary diagonals (that is, at the positions $M_{n,n\pm1}$) are the coupling energies between two oscillators n and $n \pm 1$. A normal mode of the system corresponds to an *eigenvector* of the matrix $\hat{M}$, and the natural frequency of this mode corresponds to the respective *eigenvalue*.

The equation (1.110) has non-trivial solutions only, when the determinant of the matrix $\hat{M} - \omega^2$ vanishes. The eigenvalues are those ω^2 which satisfy this requirement,

$$\det(\hat{M} - \omega^2 \mathbf{1}) = 0 \ . \tag{1.111}$$

1.4.4 Dissipation in coupled oscillator systems

We now extend the system of two coupled pendulums to include damping. Assuming that the movement of the pendulum is subject to damping,

$$\begin{aligned} \ddot{\theta}_1 &= -\Gamma\dot{\theta}_1 - \tfrac{g}{L}\theta_1 - \tfrac{k}{m}(\theta_1 - \theta_2) \\ \ddot{\theta}_2 &= -\Gamma\dot{\theta}_2 - \tfrac{g}{L}\theta_2 - \tfrac{k}{m}(\theta_2 - \theta_1) \ , \end{aligned} \tag{1.112}$$

giving the collective modes,

$$\begin{aligned} \ddot{\Psi} &= \ddot{\theta}_1 + \ddot{\theta}_2 = -\Gamma\dot{\Psi} - \tfrac{g}{L}\Psi \\ \ddot{\aleph} &= \ddot{\theta}_1 - \ddot{\theta}_2 = -\Gamma\dot{\aleph} - \left(\tfrac{g}{L} + \tfrac{2k}{m}\right)\aleph \ . \end{aligned} \tag{1.113}$$

Assuming that the movement of the spring (not the movement of the pendulums) is subject to damping,

$$\begin{aligned} \ddot{\theta}_1 &= -\tfrac{g}{L}\theta_1 - \tfrac{k}{m}(\theta_1 - \theta_2) - \Gamma(\dot{\theta}_1 - \dot{\theta}_2) \\ \ddot{\theta}_2 &= -\tfrac{g}{L}\theta_2 - \tfrac{k}{m}(\theta_2 - \theta_1) - \Gamma(\dot{\theta}_2 - \dot{\theta}_1) \ , \end{aligned} \tag{1.114}$$

giving the collective modes

$$\begin{aligned} \ddot{\Psi} &= \ddot{\theta}_1 + \ddot{\theta}_2 = -\tfrac{g}{L}\Psi \\ \ddot{\aleph} &= \ddot{\theta}_1 - \ddot{\theta}_2 = -\left(\tfrac{g}{L} + \tfrac{2k}{m}\right)\aleph - 2\Gamma\dot{\aleph} \ . \end{aligned} \tag{1.115}$$

Thus, the anti-symmetric mode Ψ is free from damping, while the symmetric mode $\aleph$ damps out twice as fast. Therefore, Ψ is called the *subradiant* mode and $\aleph$ the *superradiant* mode.

1.4.5 Exercises

1.4.5.1 Ex: Energy of normal modes

Verify that the total energy of a system of two coupled oscillators is equal to the sum of the energies of the normal modes.

1.4.5.2 Ex: Normal modes of two spring-coupled masses

Consider two different masses m_1 and m_2 coupled by a spring k.
a. Determine the equation of motion and the characteristic equation for each mass.
b. Write the characteristic equations in matrix form: $\hat{M}\vec{a} = \omega^2\vec{a}$, where $\vec{a} \equiv (a_1, a_2)$

and a_j are the amplitude of the oscillations and calculate the two eigenvalues of the matrix.
c. Calculate the normal modes, that is, the eigenvectors solving the equation $\hat{M}\vec{a} = \omega_k^2\vec{a}$ for each eigenvalue.
d. Derive the differential equations of the center-of-mass motion and the relative motion. Compare the result with the normal modes.

1.4.5.3 Ex: Spring-coupled chain of masses

Consider a chain of spring-coupled masses.
a. Determine the equation of motion and the characteristic equation for each mass.
b. Calculate the normal modes for a chain consisting of three masses.

1.4.5.4 Ex: Normal modes of CO_2

We consider the carbon dioxide molecule CO_2, for which we make a spring-mass model with three masses coupled by k springs in a linear chain. Calculate the frequencies of the normal modes and the eigenvectors of the vibrations.

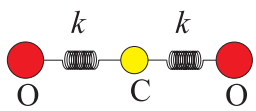

Figure 1.42: Normal modes of CO_2.

1.4.5.5 Ex: Three coupled pendulums

Determine the frequencies of the oscillation modes of a chain of three spring-coupled pendulums.

1.4.5.6 Ex: Super- and subradiance

We consider three carts attached by springs (spring constant k), as shown in the figure. The inner carts have mass m and are subject to damping by friction with the coefficient γ. The outer cart has mass M and friction Γ.
a. Establish the equations of motion of the three carts.
b. Discuss the case $M \to 0$.

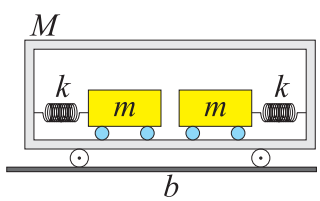

Figure 1.43: Super- and subradiant pendulums.

1.5 Further reading

H.M. Nussenzveig, Edgar Blucher (2014), *Curso de Física Básica: Fluidos, Vibrações e Ondas, Calor - vol 2* [ISBN]

Chapter 2

Waves

While in *vibrating* bodies the motion and the energy are localized in space, *waves* do propagate and carry energy to other places. In fact, waves represent the most important mechanism for transporting and exchanging energy and information. We can understand a wave as a perturbation propagating through an elastic material medium. In some cases, however, e.g. for electromagnetic waves, the propagation of the wave is due to a self-sustained oscillation between two forms of energy (electric and magnetic) without the need of a material medium. Here, is a classification of the most common types of waves: A lecture version of this chapter can be found at

Table 2.1: *Types of waves.*

wave	pulse	sound	sound	surface	light	de Broglie
medium	string	air	crystal	fluid	vacuum	particle
polarize	trans.	long.	trans./long.	long.	trans.	long.
transform	Galilei	Galilei	Galilei	Galilei	Lorentz	Galilei
wave eq.	Helmholtz	Helmholtz	Helmholtz	Helmholtz	Helmholtz	Schrödinger

(watch talk).

2.1 Propagation of waves

There are several types of wave that we will classify according to the propagation medium and to the polarization, that is, we will distinguish longitudinal and transverse waves. There are media only supporting transverse waves (strings, water surfaces). Others only withstand longitudinal waves (sound in fluid media). Finally, there are media supporting both (sound in solids, electromagnetic waves).

The simplest example of a pulse is a local deformation of a string, as shown in Fig. 2.1. The pulse travels to one end of the string by a motion called *propagation*. The propagation is not conditioned to any transport of mass, but all the particles of the system go back to their original positions after the passage of the pulse. However, there is energy transport along the string, since each of its portions suffers an increase in kinetic and potential energy during the passage of the pulse.

In general, the pulse broadens during propagation, an effect called *dispersion*. To simplify the problem let us, as a first approximation neglect the dispersion and

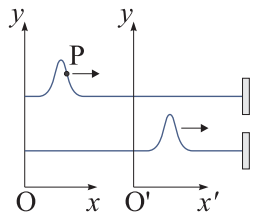

Figure 2.1: Pulse propagation along a rope.

suppose that the pulse does not change its shape,

$$Y(x,t) = f(x - vt) \ , \tag{2.1}$$

where the propagation velocity is positive when the pulse propagates in the direction of the positive x-axis.

The behavior of the pulse at the end of the rope depends on its fixation. Attached to a wall, the reflected pulse has opposite propagation amplitude and direction,

$$Y_{rfl}(x,t) = -f(x + vt) \ . \tag{2.2}$$

Fixed to another rope, the pulse will be partially reflected and partially transmitted.

2.1.1 Transverse waves, propagation of pulses on a rope

Pulses on a rope are examples for transverse waves. The speed at which the pulse propagates on a rope depends essentially on the properties of the string, that is, its mass density μ and the applied tension T, but not on the pulse amplitude. We take a small length element dx of the string with mass $dm = \mu dx$ and consider a pulse traveling with velocity v, as shown in Fig. 2.1.

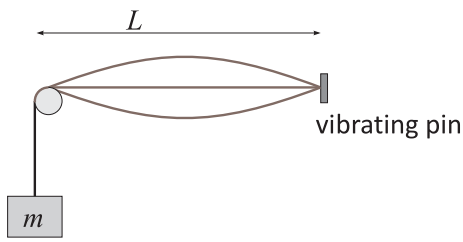

Figure 2.2: Mass element of a rope upon a passage of a pulse.

The vertical force due to the difference of tensions is,

$$F_y = T \sin\theta(x + dx) - T \sin\theta(x) \ . \tag{2.3}$$

Assuming $\theta(x)$ small, such that $\sin\theta(x) \simeq \tan\theta(x) = \frac{dY}{dx}$,

$$F_y = T \left(\frac{dY}{dx}\right)_{x+dx} - T \left(\frac{dY}{dx}\right)_x = T \frac{\partial^2 Y}{\partial x^2} dx \ . \tag{2.4}$$

On the other hand, applying Newton's second law to this string element, we find,

$$F_y = dm \frac{\partial^2 Y}{\partial t^2} \ . \tag{2.5}$$

Thus,

$$\frac{\partial^2 Y}{\partial x^2} = \frac{\mu}{T} \frac{\partial^2 Y}{\partial t^2} \ . \tag{2.6}$$

This equation is called *wave equation* and fully describes the propagation of the pulse on the string. Since $Y = f(x - vt)$ depends on both x and t, the derivatives that appear in the equation are partial, that is, one derives with respect to one variable keeping the other constant. To find the velocity, we write,

$$\frac{\partial^2 Y}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial x}{\partial t}\frac{\partial Y}{\partial x}\right) = v\frac{\partial}{\partial t}\left(\frac{\partial Y}{\partial x}\right) = v\frac{\partial}{\partial x}\left(\frac{\partial Y}{\partial t}\right) = v\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial t}\frac{\partial Y}{\partial x}\right) = v^2\frac{\partial Y^2}{\partial x^2} \ , \tag{2.7}$$

and compare the second relation with the wave equation, finding,

$$v = \sqrt{\frac{T}{\mu}} \ . \tag{2.8}$$

Example 10 (*Reflection of pulses on a rope*):

- Excite a pulse on a rope fixed to the wall (i) directly or (ii) through a thinner rope.

2.1.2 Longitudinal waves, propagation of sonar pulses in a tube

Acoustic pulses are examples for longitudinal waves. They are due to a process of compression and decompression of a gaseous medium (such as air), liquid or even solid. Let us consider an oscillating piston inside a tube (cross section A) filled with air of mass density ρ_0, as shown in Fig. 2.3. When the piston moves, it causes a local pressure increase. We want to find the velocity v at which the compression travels along the tube.

As shown in Fig. 2.3, the piston causes a negative pressure gradient along the tube giving rise to an unbalanced force which accelerates mass elements of air to the right. To simplify the situation let us assume that the piston is moved with velocity u within a time interval Δt compressing the volume of the tube by a value

$$\Delta V = -Au\Delta t \ . \tag{2.9}$$

During this time, the piston accelerates a mass $m = \rho_0 V$ of air within a volume V given by the propagation velocity v of the pulse along the tube,

$$V = Av\Delta t \ . \tag{2.10}$$

The mass within this volume receives a momentum,

$$F\Delta t = mu \ . \tag{2.11}$$

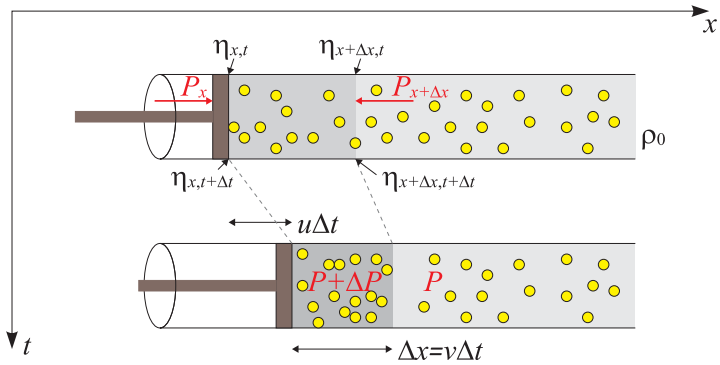

Figure 2.3: Sound waves produced by a swinging piston.

The pressure difference inside and outside the volume V causes a pressure imbalance,

$$F = A\Delta P \ . \tag{2.12}$$

With these relations we can calculate the *compressibility* of the gas,

$$\frac{1}{\kappa} \equiv -\frac{\Delta P}{\Delta V/V} = \frac{F/A}{u/v} = \frac{mu/A\Delta t}{u/v} = \frac{\rho_0 V v}{A\Delta t} = \rho_0 v^2 \ , \tag{2.13}$$

we obtain the propagation velocity of the pulse in the gas,

$$v = \sqrt{\frac{1}{\kappa\rho_0}} \ . \tag{2.14}$$

Thus, the velocity of sound propagation depends critically on the material medium. We have $v_{ar} = 331\,\mathrm{m/s}$, $v_{H_2} = 1286\,\mathrm{m/s}$, $v_{H_2O} = 331\,\mathrm{m/s}$, $v_{rubber} = 54\,\mathrm{m/s}$, and $v_{Al} = 5100\,\mathrm{m/s}$.

To derive the equation of motion, we consider a thin gas element with thickness Δx and mass $m = \rho_0 A\Delta x$ subject to a difference of pressure on both sides of,

$$P_x - P_{x+\Delta x} = -\frac{\partial P_x}{\partial x}\Delta x = -\frac{\partial}{\partial x}(P_0 + \Delta P)\Delta x = -\frac{\partial \Delta P}{\partial x}\Delta x \ , \tag{2.15}$$

where we subtracted the background pressure P_0 assumed to be constant. This pressure difference creates a force $F = A(P_x - P_{x+\Delta x})$ accelerating the gas element following Newton's law, $F = m\ddot{\eta}$, where $\eta(x)$ is the displacement of the element, such that,

$$\frac{\partial \eta}{\partial x} = \frac{\Delta V}{V} \tag{2.16}$$

and the compression (see Fig. 2.3). We therefore obtain,

$$\rho_0 \Delta x \frac{\partial^2 \eta}{\partial t^2} = \frac{F}{A} = -\frac{\partial \Delta P}{\partial x}\Delta x \ . \tag{2.17}$$

Substituting ΔP by the relationship (2.13),

$$\rho_0 \frac{\partial^2 \eta}{\partial t^2} = -\frac{\partial}{\partial x}\left(-\frac{1}{\kappa}\frac{\partial \eta}{\partial x}\right) = \frac{1}{\kappa}\frac{\partial^2 \eta}{\partial x^2} \ , \tag{2.18}$$

which gives the wave equation. Solve Excs. 2.1.7.1, 2.1.7.2, and 2.1.7.3.

2.1.3 Electromagnetic waves

Electromagnetic waves are in several aspects different from mechanical longitudinal or transverse waves. For example, they do not need a propagation medium, but move through the vacuum at an extremely high speed. The speed of light, $c = 299792458\,\mathrm{m/s}$ exactly, is so high, that the laws of classical mechanics are no longer valid, but must be replaced by relativistic laws. And since there is no propagation medium, with respect to vacuum all inertial systems are equivalent, which will have important consequences for the Doppler effect. We will show that the electromagnetic wave equation almost comes out as a corollary of the theory of special relativity.

Electromagnetic waves always arise when a charge changes position. In this way the theory of electromagnetic waves is also a consequence of the theory electromagnetism, which is contained in Maxwell's equations. We will introduce here, without derivation, the wave equation for the electric and magnetic fields.

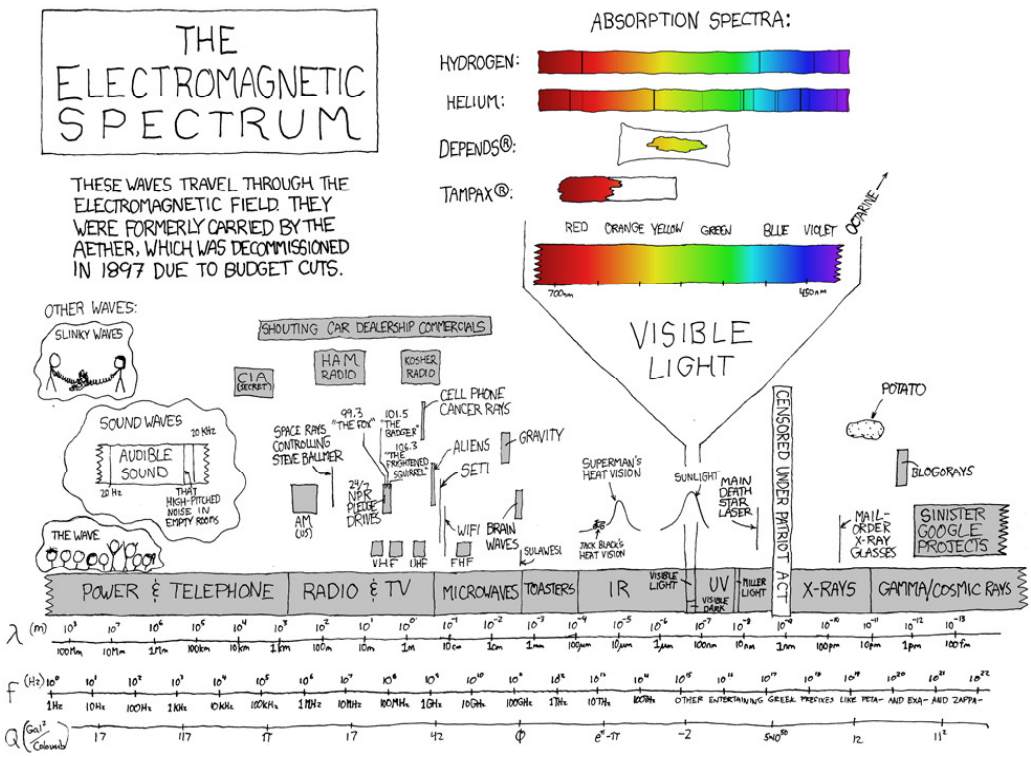

Figure 2.4: The electromagnetic spectrum.

2.1.3.1 Helmholtz equation

We have already seen how the periodic conversion between kinetic and potential energy in a pendulum can propagate in space when the pendulum is coupled to other pendulums attached to each other in a chain, and that this model explains the propagation of a pulse on the string. We also discussed how electrical and magnetic

energy can be interconverted in an electronic L-C-circuit with a capacitor storing electrical energy and an inductance (a coil) storing magnetic energy. The law of electrodynamics describing the transformation of electric field variations into magnetic energy is *Ampère's law*, and the law describing the transformation of magnetic field variations into electric energy is *Faraday's law*,

$$\frac{\partial \vec{\mathcal{E}}}{\partial t} \curvearrowright \vec{\mathcal{B}}(t) \quad , \quad \frac{\partial \vec{\mathcal{B}}}{\partial t} \curvearrowright -\vec{\mathcal{E}}(t) \ . \tag{2.19}$$

Extending the circuit L-C to a chain, it is possible to show that the electromagnetic oscillation propagates along the chain. This model describes well the propagation of electromagnetic energy along a coaxial cable or the propagation of light in free space.

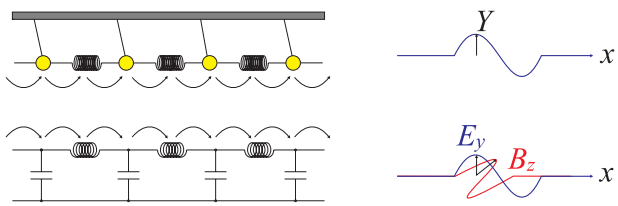

Figure 2.5: Analogy between the propagation of mechanical waves (above) and electromagnetic waves (below).

The *electrical energy* stored in the capacitor and the *magnetic energy* stored in the coil are given by,

$$E_{ele} = \tfrac{\varepsilon_0}{2}|\vec{\mathcal{E}}|^2 \quad , \quad E_{mag} = \tfrac{1}{2\mu_0}|\vec{\mathcal{B}}|^2 \ , \tag{2.20}$$

where the constants $\varepsilon_0 = 8.854 \cdot 10^{-12}$ As/Vm and $\mu_0 = 4\pi \cdot 10^{-7}$ Vs/Am are called *permittivity* and *permeability* of the vacuum. By analogy with the waves on a string, we can write the wave equations (called *Helmholtz equations*) for plane electromagnetic waves propagating along the x-axis,

$$\frac{\partial^2 \mathcal{E}_y}{\partial t^2} = \frac{1}{\varepsilon_0 \mu_0}\frac{\partial^2 \mathcal{E}_y}{\partial x^2} \quad , \quad \frac{\partial^2 \mathcal{B}_z}{\partial t^2} = \frac{1}{\varepsilon_0 \mu_0}\frac{\partial^2 \mathcal{B}_z}{\partial x^2} \ . \tag{2.21}$$

The formal derivation must be made from *Maxwell's equations*, which are the fundamental equations of the theory of electrodynamics. Here, we only note that,

- electromagnetic waves (in free space) are transverse;
- the electric field vector, the magnetic field vector, and the direction of propagation are orthogonal;
- the propagation velocity is the speed of light, because $c^2 = 1/\varepsilon_0\mu_0$.

2.1.3.2 Radiation intensity

In electrodynamic theory the energy flux is calculated by the *Poynting vector*,

$$\vec{S}(\mathbf{r}, t) = \tfrac{1}{\mu_0}\vec{\mathcal{E}}(\mathbf{r}, t) \times \vec{\mathcal{B}}(\mathbf{r}, t) \ . \tag{2.22}$$

The absolute value is the intensity of the light field,

$$I(\mathbf{r}, t) = |\mathbf{S}(\mathbf{r}, t)| \ . \tag{2.23}$$

2.1.4 Harmonic waves

In general, a light field is a superposition of many waves with many different frequencies and polarizations and propagating in many directions. The laser is an exception. Being monochromatic, polarized, directional, and coherent, it is very close to the ideal of an *harmonic wave*, that is, a wave described by the function,

$$Y(x, t) = Y_0 \cos(kx - \omega_0 t) \ , \tag{2.24}$$

where $\omega_0 = 2\pi\nu$ is the angular frequency of the oscillation and $k = 2\pi/\lambda$ the wavevector. By inserting this function into the *wave equation*,

$$\frac{\partial^2 Y}{\partial t^2} = c^2 \frac{\partial Y^2}{\partial x^2} \ , \tag{2.25}$$

where we now call c the propagation velocity of the harmonic wave, we verify the *dispersion relation*,

$$\omega = ck \ . \tag{2.26}$$

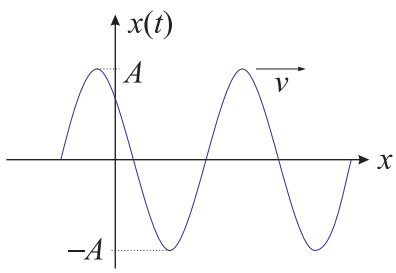

Figure 2.6: Illustration of a harmonic wave.

Often, the propagation velocity is independent of the wavelength, $c(k) = const$. In this case, a wave composed of several waves with different wavevectors k propagates without dispersing, that is, without changing its shape. In other cases, when $c(k) \neq const$, the wave deforms along its path.

2.1.5 Wave packets

Since the wave equation (2.25) is linear, the superposition principle is valid, that is, if Y_1 and Y_2 are solutions, then $\alpha Y_1 + \beta Y_2$ also is. More generally, we can say that, if $A(k)e^{\imath(kx-\omega t)}$ is a solution satisfying the wave equation for any k, then obviously,

$$Y(x, t) = \int_{-\infty}^{\infty} A(k) e^{\imath(kx-\omega t)} dk \ , \tag{2.27}$$

is. This means that the displacement $Y(x)$ and the distribution of amplitudes $A(k)$ are related by Fourier transform, $Y(x,t) = e^{-i\omega t}\mathcal{F}A(k)$.

Assuming a Gaussian distribution of wavevectors characterized by the width [1] Δk, $A(k) = e^{-(k-k_0)^2/2\Delta k^2}$, we obtain as solution for the wave equation,

$$\begin{aligned} Y(x,t) &= \int_{-\infty}^{\infty} e^{-(k-k_0)^2/2\Delta k^2} e^{\imath(kx-\omega t)} dk \\ &= e^{\imath(k_0 x-\omega t)} \int_{-\infty}^{\infty} e^{-q^2/2\Delta k^2} e^{\imath q x} dq = \sqrt{2\pi k} e^{-\Delta k^2 x^2/2} e^{\imath(k_0 x-\omega t)} \ . \end{aligned} \tag{2.28}$$

This solution of the wave equation describes an *wave packet* with a Gaussian envelope [2], that is, a localized perturbation, as we discussed at the initial example of a pulse propagating on a string. Obviously, other distributions of wavevectors are possible.

Note that the width of the distribution of wavevectors, Δk, and that of the spatial distribution, $\Delta x \equiv 1/\Delta k$ satisfy a relation called *Fourier's theorem*,

$$\Delta x \Delta k = 1 \ , \tag{2.29}$$

which in quantum mechanics turns into *Heisenberg's uncertainty relation*: The broader a wavevector distribution, the narrower the spatial distribution, and vice versa. In the limit of a sinusoidal wave described by a single wavevector, we expect a infinite spatial extension of the wave.

2.1.6 Dispersion

We consider a superposition of two waves,

$$\begin{aligned} Y_1(x,t) + Y_2(x,t) &= a\cos(k_1 x - \omega_1 t) + a\cos(k_2 x - \omega_2 t) \\ &= 2a\cos\left[\tfrac{(k_1-k_2)x}{2} - \tfrac{(\omega_1-\omega_2)t}{2}\right]\cos\left[\tfrac{(k_1+k_2)x}{2} - \tfrac{(\omega_1+\omega_2)t}{2}\right] \ . \end{aligned} \tag{2.30}$$

The resulting wave can be regarded as a wave of frequency $\frac{1}{2}(\omega_1+\omega_2)t$ and wavelength $\frac{1}{2}(k_1+k_2)$, whose amplitude is modulated by an envelope of frequency $\frac{1}{2}(\omega_1-\omega_2)t$ and wavelength $\frac{1}{2}(k_1-k_2)x$.

In the absence of dispersion the *phase velocities* of the two waves and the propagation velocity of the envelope, called *group velocity*, are equal,

$$c = \frac{\omega_1}{k_1} = \frac{\omega_2}{k_2} = \frac{\omega_1-\omega_2}{k_1-k_2} = \frac{\Delta\omega}{\Delta k} = v_g \ . \tag{2.31}$$

[1] Δk is half the *total* Gaussian width at *rms* (root-mean-square) height, that is, at $1/\sqrt{e}$ of the maximum.

[2] The definition of the Fourier transform in one dimension is,

$$Y(x) = \mathcal{F}A(k) \equiv \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{\imath kx} dk \ .$$

For the Gaussian function we have,

$$\begin{aligned} Y(x) &= \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ak^2} e^{\imath kx} dk = \tfrac{1}{\sqrt{2\pi}} e^{-x^2/4a} \int_{-\infty}^{\infty} e^{-a(k-ix/2a)^2} dk \\ &= \tfrac{1}{\sqrt{2\pi}} e^{-x^2/4a} \int_{-\infty}^{\infty} e^{-aq^2} dq = \tfrac{1}{\sqrt{2a}} e^{-x^2/4a} \ . \end{aligned}$$

However, the phase velocities of the two harmonic waves can also be different, such that the frequency depends on the wavelength, $\omega = \omega(k)$. In this case, the phase velocity also varies with the wavelength,

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}(kc) = c + k\frac{dc}{dk} \ . \tag{2.32}$$

Often this variation is not very strong, such that it is possible to expand,

$$\omega(k) = \omega_0 + \left.\frac{d\omega}{dk}\right|_{k_0} \cdot (k-k_0) + \frac{1}{2}\left.\frac{d^2\omega}{dk^2}\right|_{k_0} \cdot (k-k_0)^2 \equiv \omega_0 + v_g(k-k_0) + \beta(k-k_0)^2 \ . \tag{2.33}$$

In general we have, $v_g < c$, a situation that is called *normal dispersion*. But there are examples of *abnormal dispersion*, where $v_g > c$, e.g. close to resonances or with matter waves characterized by a quadratic dispersion relation $\hbar\omega = (\hbar k)^2/2m$.

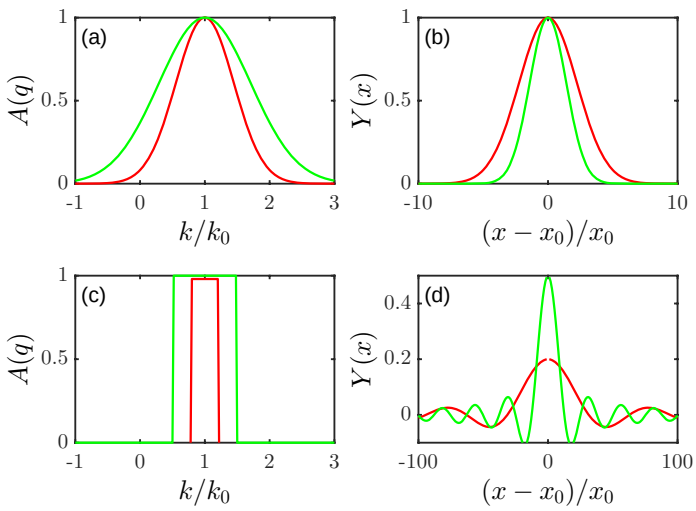

Figure 2.7: (code) Gaussian (upper graphs) and rectangular (lower graphs) distribution of amplitudes in momentum space (left) and in position space (right).

2.1.6.1 Rectangular wave packet with linear dispersion

As an example, we determine the shape of the wavepacket for a rectangular amplitude distribution, $A(k) = A_0\chi_{[k_0-\Delta k/2, k_0+\Delta k/2]}$, subject to linear dispersion (expansion up

to the linear term in Eq. (2.33)). By the Fourier theorem,

$$
\begin{aligned}
Y(x,t) &= \int_{-\infty}^{\infty} A(k) e^{\imath(kx-\omega t)} dk = A_0 \int_{k_0-\Delta k/2}^{k_0+\Delta k/2} e^{\imath(kx-\omega_0 t + \frac{d\omega}{dk}\big|_{k_0}(k-k_0)t)} dk \qquad (2.34)\\
&= A_0 e^{\imath(k_0 x-\omega_0 t)} \int_{k_0-\Delta k/2}^{k_0+\Delta k/2} e^{\imath(k-k_0)\left(x-\frac{d\omega}{dk}\big|_{k_0} t\right)} dk\\
&= A_0 e^{\imath(k_0 x-\omega_0 t)} \int_{-\Delta k/2}^{\Delta k/2} e^{\imath k\left(x-\frac{d\omega}{dk}\big|_{k_0} t\right)} dk = A_0 e^{\imath(k_0 x-\omega_0 t)} \int_{-\Delta k/2}^{\Delta k/2} e^{\imath k u} dk\\
&= A_0 e^{\imath(k_0 x-\omega_0 t)} \frac{e^{\imath \Delta k/2 u} - e^{-i\Delta k/2 u}}{\imath u} = 2A_0 e^{\imath(k_0 x-\omega_0 t)} \frac{\sin \frac{u\Delta k}{2}}{u} \equiv A(x,t) e^{\imath(k_0 x-\omega_0 t)} \ .
\end{aligned}
$$

With the abbreviation $u \equiv x - \frac{d\omega}{dk}\big|_{k_0} t = x - v_g t$ the interpretation of the group velocity becomes obvious,

$$
v_g \equiv \left.\frac{d\omega}{dk}\right|_{k_0} t \ . \qquad (2.35)
$$

The envelope has the shape of a 'sinc' function, such that the intensity of the wave is,

$$
|Y(x,t)|^2 = A_0 \Delta k \ \mathrm{sinc}\left[\tfrac{\Delta k}{2}(x - v_g t)\right] \ . \qquad (2.36)
$$

Obviously, the *wavepacket* is localized in space. It moves at group velocity, but does not diffuse.

2.1.6.2 Dispersion of a Gaussian wave packet subject to quadratic dispersion

Quadratic dispersion leads to a spreading of the wavepackets. We show this at the example of the Gaussian wavepacket $A(k) = e^{-\alpha(k-k_0)^2}$, expanding the dispersion relation (2.33) up to the quadratic term. By the Fourier theorem,

$$
\begin{aligned}
Y(x,t) &= \int_{-\infty}^{\infty} A(k) e^{\imath(kx-\omega t)} dk = A_0 e^{\imath(k_0 x-\omega_0 t)} \int_{-\infty}^{\infty} e^{\imath(k-k_0)(x-v_g t) - (\alpha+i\beta t)(k-k_0)^2} dk\\
&= A_0 e^{\imath(k_0 x-\omega_0 t)} \int_{-\infty}^{\infty} e^{\imath k(x-v_g t) - (\alpha+i\beta t)k^2} dk\\
&\equiv A_0 e^{\imath(k_0 x-\omega_0 t)} \int_{-\infty}^{\infty} e^{\imath k u - v k^2} dk = A_0 \sqrt{\tfrac{\pi}{v}} e^{\imath(k_0 x-\omega_0 t)} e^{-u^2/4v} \ . \qquad (2.37)
\end{aligned}
$$

The absolute square of this solution describes the spatial energy distribution of the wavepacket,

$$
|Y(x,t)|^2 = A_0^2 \frac{\pi}{\sqrt{vv^*}} e^{-u^2/4v - u^2/4v^*} = A_0^2 \frac{\pi}{x_0\sqrt{\alpha/2}} e^{-(x-v_g t)^2/x_0^2} \ , \qquad (2.38)
$$

with $x_0 \equiv \sqrt{2\alpha}\sqrt{1+\frac{\beta^2}{\alpha^2}t^2}$. Obviously, for long times the pulse spreads out at constant speed. Since the constant α gives the initial width of the pulse, we realize that an initially compressed pulse spreads faster. Therefore, the angular coefficient of the dispersion relation determines the group velocity, while the curvature determines the spreading speed (dispersion).

2.1.7 Exercises

2.1.7.1 Ex: Speed of sound

A person drops a stone from the top of a bridge and hears the sound of the stone hitting the water after $t = 4\,\mathrm{s}$.
a. Estimate the distance between the bridge and the water level, assuming that the propagation time of sound is negligible.
b. Improve the estimate by taking into account the finite speed of sound.

2.1.7.2 Ex: Distance of a lightning

An approximate method for estimating the distance of a lightning consists in starting to count the seconds when the lightning stroke and stop counting when the thunder arrives. The number of seconds counted divided by 3 gives the distance from the lightning in kilometers. Estimate accuracy of this procedure.

2.1.7.3 Ex: Speed of sound

A student in her room listens to the radio broadcasting a nearby football game. She is 1.6 km south of the field. On the radio, the student hears the noise generated by an electromagnetic pulse caused by a lightning strike. Two seconds later she hears the noise of thunder on the radio which was captured by the microphone of the football field. Four seconds after hearing the noise on the radio, she hears the noise of the thunder directly. Where did the lightning strike in relation to the soccer field?

2.1.7.4 Ex: Absence of dispersion in sound

Discuss the experimental evidence that leads us to assume that the speed of sound in the audible range must be the same at all wavelengths.

2.1.7.5 Ex: Optical dispersion

a. While vacuum is strictly dispersionless, the refractive index of air depends on the wavelength of light λ, on temperature T in $^\circ$ C and on the atmospheric pressure P in mbar like,

$$n_s = 1 + 10^{-8}\left(8342.13 + \frac{2406030}{130 - 10^{12}/\lambda^2} + \frac{15997}{38.9 - 10^{12}/\lambda^2}\right)$$
$$n = 1 + (n_s - 1)\frac{0.00185097P}{1 + 0.003661T}\ .$$

Calculate the dispersion of air within range $\lambda_1 = 400\,\mathrm{nm}$ and $\lambda_2 = 800\,\mathrm{nm}$.
b. Using Snell's law,

$$\frac{n_1}{n_2} = \frac{\sin\alpha_2}{\alpha_1}\ ,$$

calculate the angular dispersion $d\alpha_{ar}/d\lambda$ of a beam of light at the interface between vacuum and atmospheric air for $P = 1013\,\mathrm{mbar}$ and $T = 25^\circ$ C around $\lambda = 500\,\mathrm{nm}$.

2.1.7.6 Ex: Dispersion near an atomic resonance

Near an atomic resonance ω_0 the refractive index can be approximated by,

$$n = 1 - \frac{\alpha}{\omega^2 - \omega_0^2} \ ,$$

where the polarizability of the gas α is a constant. Calculate the group velocity $v_g(\omega_l)$ of a laser wave packet passing through a gas of these atoms as a function of the laser frequency ω_l. Approximate $|\omega - \omega_0| \ll \omega_0$. Make a qualitative chart of $n(\omega_l)$, $k(\omega_l)$, of the phase velocity $v_f(\omega_l)$, and of the group velocity $v_g(\omega_l)$.

2.1.7.7 Ex: Group velocity near a broad transition

The refractive index of a dilute gas (density ρ) of atoms excited by a light beam of frequency ω near a transition (resonant frequency ω_0 and width Γ) can be approximated by,

$$n = \sqrt{1 - \frac{4\pi\rho\Gamma}{k_0^3(2\Delta + \imath\Gamma)}} \simeq 1 - \frac{2\pi\rho\Gamma}{k_0^3(2\Delta + \imath\Gamma)} \ ,$$

where $ck_0 = \omega_0$ and $\Delta \equiv \omega - \omega_0$. Calculate the group velocity near resonance.

2.1.7.8 Ex: Dispersion in a metal

The dispersion ratio in metals can be approximated by,

$$n^2(\omega) = 1 + \omega_p^2 \left(\frac{f_e}{-\omega^2 - \imath\gamma_e\omega} + \sum_j \frac{f_j}{\omega_{0j}^2 - \omega^2 - \imath\gamma_j\omega} \right) \ ,$$

where ω_p is called the plasma frequency and f_e and f_j are constants. Calculate the group velocity $v_g(\omega)$.

2.2 The Doppler effect

2.2.1 Sonic Doppler effect

Waves propagate from a source to an listener within an elastic material medium with the propagation velocity v. So far, we assumed the source, the medium, and the listener at rest. The question now is, what happens when one of these three components gets in motion.

2.2.1.1 Source in motion

We imagine a source emitting signals at frequency f_0. Within the time of a period $T = \frac{1}{f_0}$ these pulses travel a distance,

$$\lambda = vT = \frac{v}{f_0} \ , \tag{2.39}$$

within the medium. While the source is at rest, the distance between the pulses is λ. However, when the source moves in the propagation direction of the pulses, a resting listener judges that the pulses are emitted within the medium at reduced distances Δx, as shown in Fig. 2.8,

$$\Delta x = \lambda - u_s T \ . \tag{2.40}$$

A listener now receives the pulses at the increased frequency of,

$$f = \frac{v}{\Delta x} = \frac{v}{\lambda - u_s T} = \frac{v f_0}{v - u_s} = \frac{f_0}{1 - u_s/v} \ . \tag{2.41}$$

This effect is called *sonic Doppler effect*. For small velocities we can expand,

$$f = \frac{f_0}{1 \mp u_s/v} \simeq f_0 \left(1 \pm \frac{u_s}{v}\right) \ , \tag{2.42}$$

where the upper (lower) signals apply, when the source approaches (moves away from) the listener.

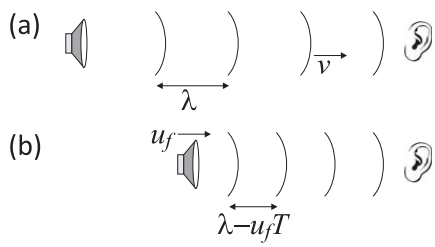

Figure 2.8: Doppler effect due to a motion of the source. In (a) the source is at rest, in (b) it moves toward the listener.

2.2.1.2 Listener in motion

Again, we consider the same source emitting signals at frequency f_0. While the source is at rest, the distance between the pulses is λ. However, when the listener is approaching the source, as shown in Fig. 2.9, pulses are recorded by the listener in a shorter time intervals,

$$T = \frac{\lambda}{v + u_r} = \frac{1}{f} \ . \tag{2.43}$$

That is, the listener measures a larger number of pulses,

$$f = f_0 \left(1 \pm \frac{u_r}{v}\right) \ , \tag{2.44}$$

where the upper (lower) signs apply, when the receiver approaches (moves away from) the source.

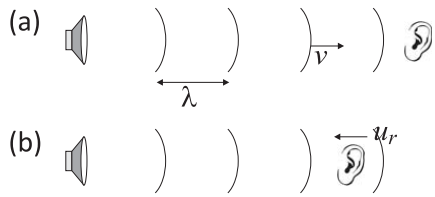

Figure 2.9: Doppler effect due to a motion of the listener. In (a) the listener is at rest, in (b) it moves toward the source.

2.2.1.3 Moving medium

We can combine the two Doppler effects into a single expression,

$$f = f_0 \frac{v^2 - \mathbf{v} \cdot \mathbf{u}_r}{v^2 - \mathbf{v} \cdot \mathbf{u}_s} \ . \tag{2.45}$$

The cases discussed above refer to the source or the listener being in motion *with respect to the medium carrying the wave* considered at rest. If the medium is moving at a velocity u_m, e.g. due to a wind moving the air, the velocities of the source and the listener with respect to the medium are modified, $u_s \to u_s - u_m$ and $u_r \to u_r - u_m$, such that,

$$f = f_0 \frac{1 - (u_r - u_m)/v}{1 - (u_s - u_m)/v} \ . \tag{2.46}$$

The same result is obtained by a transformation of the propagation velocity of the sound, $v \to v + u_m$.

2.2.2 Wave equation under Galilei transformation

The Galilei transformation says, that we obtain the function describing the motion in the system S' simply by substituting $x \to x'$ and $t \to t'$ with [3],

$$\begin{aligned} t' \equiv t \quad &\text{and} \quad x' \equiv x - ut \quad \text{or} \\ t \equiv t' \quad &\text{and} \quad x \equiv x' + ut \ , \end{aligned} \tag{2.47}$$

which implies,

$$v' = \frac{\partial x'}{\partial t'} = \frac{\partial x}{\partial t} - u = v - u \ . \tag{2.48}$$

Newton's classical mechanics is *Galilei invariant*, which means that fundamental equations of the type,

$$m\dot{v}_i = -\nabla_{x_i} \sum_j V_{ij}(|x_i - x_j|) \ , \tag{2.49}$$

[3]Note that the Galilei transform,

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = G \begin{pmatrix} ct \\ x \end{pmatrix} \qquad \text{with} \qquad G \equiv \begin{pmatrix} 1 & 0 \\ -\beta & 1 \end{pmatrix}$$

is unitary because $\det G = 1$.

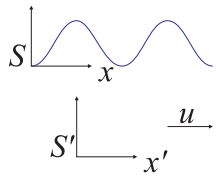

Figure 2.10: Wave in the inertial system S seen by an observer moving at the velocity u in the system S'.

do not change their form under *Galilei transform*. In contrast, the wave equation is not Galilei invariant. To see this, we consider a wave in the inertial system S being at rest with respect to the propagation medium. The wave is described by $Y(x,t)$ and satisfies the wave equation,

$$\frac{\partial^2 Y(x,t)}{\partial t^2} = c^2 \frac{\partial^2 Y(x,t)}{\partial x^2} \ . \tag{2.50}$$

An observer be in the inertial system S' moving with respect to S with velocity u, such that $x' = x - ut$. The question now is, what is the equation of motion for this wave described by $Y'(x',t')$, that is, we want to check the validity of,

$$\frac{\partial^2 Y'(x',t')}{\partial t'^2} \stackrel{?}{=} c^2 \frac{\partial^2 Y'(x',t')}{\partial x'^2} \ . \tag{2.51}$$

For example, the wave $Y(x,t) = \sin k(x - ct)$ traveling to the right is perceived in the system S', also traveling to the right, as $Y'(x',t') = \sin k[x' - (c-u)t'] = Y(x,t)$. Hence,

$$Y'(x',t') = Y(x,t) \ , \tag{2.52}$$

that is, we expect that the laws valid in S are also valid in S'. We calculate the partial derivatives,

$$\begin{aligned}
\frac{\partial Y'(x',t')}{\partial t'} &= \frac{\partial Y(x,t)}{\partial t'} = \frac{\partial t}{\partial t'} \left.\frac{\partial Y(x,t)}{\partial t}\right|_{x=const} + \frac{\partial x}{\partial t'} \left.\frac{\partial Y(x,t)}{\partial x}\right|_{t=const} = \frac{\partial Y(x,t)}{\partial t} + u \frac{\partial Y(x,t)}{\partial x} \\
\frac{\partial Y'(x',t')}{\partial x'} &= \frac{\partial Y(x,t)}{\partial x'} = \frac{\partial t}{\partial x'} \left.\frac{\partial Y(x,t)}{\partial t}\right|_{x=const} + \frac{\partial x}{\partial x'} \left.\frac{\partial Y(x,t)}{\partial x}\right|_{t=const} = \frac{\partial Y(x,t)}{\partial x} \ .
\end{aligned} \tag{2.53}$$

Therefore, we come to the conclusion that in the system propagating with the wave, the wave equation is modified,

$$\begin{aligned}
\frac{\partial^2 Y'(x',t')}{\partial t'^2} &= \frac{\partial^2 Y(x,t)}{\partial t^2} + 2u \frac{\partial^2 Y(x,t)}{\partial t \partial x} + u^2 \frac{\partial^2 Y(x,t)}{\partial x^2} \\
&= c^2 \frac{\partial^2 Y(x,t)}{\partial x^2} + 2u \frac{\partial^2 Y(x,t)}{\partial t \partial x} + u^2 \frac{\partial^2 Y(x,t)}{\partial x^2} \\
&= (c^2 + u^2) \frac{\partial^2 Y(x,t)}{\partial x^2} + 2u \frac{\partial^2 Y(x,t)}{\partial t \partial x} = (c^2 - u^2) \frac{\partial^2 Y'(x',t')}{\partial x'^2} + 2u \frac{\partial^2 Y'(x',t')}{\partial t' \partial x'} \ .
\end{aligned} \tag{2.54}$$

Only in cases, where the wavefunction can be written as $Y(x,t) = f(x - ct) = f(x' - (c-u)t') = f'(x' - ct') = Y'(x',t')$, do we obtain a wave equation similar to

the one of the system S, but with the modified propagation velocity,

$$\begin{aligned}\frac{\partial^2 f'(x'-ct')}{\partial t'^2} &= (c^2-u^2)\frac{\partial^2 f'(x'-ct')}{\partial x'^2} + 2u\frac{\partial^2 f'(x'-ct')}{\partial x'\partial t'} \\ &= (c^2-u^2)\frac{\partial^2 f'(x'-ct')}{\partial x'^2} + 2u\frac{\partial^2 f(x'-(c-u)t')}{\partial x'\partial t'} \\ &= (c^2-u^2)\frac{\partial^2 f'(x'-ct')}{\partial x'^2} - 2u(c-u)\frac{\partial^2 f'(x'-ct')}{\partial x'^2} = (c-u)^2\frac{\partial^2 f'(x'-ct')}{\partial x'^2} \ .\end{aligned} \tag{2.55}$$

The observation that the wave equation is not Galilei-invariant expresses the fact that there is a preferential system for the wave to propagate, which is simply the system in which the propagation medium is at rest. Only in this inertial system will a spherical wave propagate isotropically.

Example 11 (*Wave equation under Galilei transformation*): We now verify the validity of the wave equation in the propagating system S' using the example of a sine wave,

$$\begin{aligned}&(c^2-u^2)\frac{\partial^2 \sin k[x'-(c-u)t']}{\partial x'^2} + 2u\frac{\partial^2 \sin k[x'-(c-u)t']}{\partial x'\partial t'} \\ &= -k^2(c^2-u^2)\sin k[x'-(c-u)t'] + 2uk^2(c-u)\sin k[x'-(c-u)t'] \\ &= -k^2(c-u)^2\sin k[x'-(c-u)t'] = \frac{\partial^2 \sin k[x'-(c-u)t']}{\partial t'^2} \ .\end{aligned}$$

2.2.3 Wave equation under Lorentz transformation

The question now is, how about electromagnetic waves which, as we have already noted and as has been verified by the famous Michelson experiment, survive without any medium. If there is no propagation medium, all inertial systems should be equivalent and the wave equation should be the same in all systems, as well as the propagation velocity, i.e. the speed of light. These were the consideration of *Henry Poincaré*. To resolve the problem we need another transformation than the one of *Galileo Galilei*. Who found it first was *Hendrik Antoon Lorentz*, however the biggest intellectual challenge was to accept all the consequences that this transformation bears. It was *Albert Einstein* who accepted the challenge and created a new mechanics called *relativistic mechanics*. As the wave equation for electromagnetic waves, called the *Helmholtz equation*, is a direct consequence of Maxwell's theory, it is not surprising that the relativistic theory is not only compatible with the electrodynamic theory, but provides a deeper understanding of it.

We begin by making the ansatz of a general transformation interconnecting the temporal and spatial coordinates by four unknown parameters, γ, $\tilde{\gamma}$, β, and $\tilde{\beta}$,

$$ct = \gamma(ct' + \beta x') \qquad \text{and} \qquad x = \tilde{\gamma}(x' + \tilde{\beta}ct') \ . \tag{2.56}$$

The same calculation made for the Galilei transform now gives the first derivatives,

$$\begin{aligned}\frac{\partial Y'(x',t')}{c\partial t'} &= \frac{\partial Y(x,t)}{c\partial t'} = \frac{\partial t}{\partial t'}\left.\frac{\partial Y(x,t)}{c\partial t}\right|_{x=const} + \frac{\partial x}{c\partial t'}\left.\frac{\partial Y(x,t)}{\partial x}\right|_{t=const} = \gamma\frac{\partial Y(x,t)}{c\partial t} + \tilde{\gamma}\tilde{\beta}\frac{\partial Y(x,t)}{\partial x} \\ \frac{\partial Y'(x',t')}{\partial x'} &= \frac{\partial Y(x,t)}{\partial x'} = \frac{\partial ct}{\partial x'}\left.\frac{\partial Y(x,t)}{c\partial t}\right|_{x=const} + \frac{\partial x}{\partial x'}\left.\frac{\partial Y(x,t)}{\partial x}\right|_{t=const} = \gamma\beta\frac{\partial Y(x,t)}{c\partial t} + \tilde{\gamma}\frac{\partial Y(x,t)}{\partial x} \ .\end{aligned} \tag{2.57}$$

The second derivatives and the application of the wave equation in the system S give,

$$\begin{aligned}\frac{\partial^2 Y'(x',t')}{c^2\partial t'^2} &= \gamma^2\frac{\partial^2 Y(x,t)}{c^2\partial t^2} + 2\gamma\tilde{\gamma}\tilde{\beta}\frac{\partial^2 Y(x,t)}{c\partial t\partial x} + (\tilde{\gamma}\tilde{\beta})^2\frac{\partial^2 Y(x,t)}{\partial x^2} \\ &= \gamma^2\frac{\partial^2 Y(x,t)}{\partial x^2} + 2\gamma\tilde{\gamma}\tilde{\beta}\frac{\partial^2 Y(x,t)}{c\partial t\partial x} + (\tilde{\gamma}\tilde{\beta})^2\frac{\partial^2 Y(x,t)}{c^2\partial t^2} \\ &= (\gamma\beta)^2\frac{\partial^2 Y(x,t)}{c^2\partial t^2} + 2\gamma\tilde{\gamma}\beta\frac{\partial^2 Y(x,t)}{c\partial t\partial x} + \tilde{\gamma}^2\frac{\partial^2 Y(x,t)}{\partial x^2} = \frac{\partial^2 Y(x',t')}{\partial x'^2} \ .\end{aligned} \tag{2.58}$$

That is, the wave equation in the system S' has the same form [4]. Thus, the requirement of invariance of the wave equation allows to affirm,

$$\gamma = \tilde{\gamma} \quad \text{and} \quad (\gamma\beta)^2 = (\tilde{\gamma}\tilde{\beta})^2 \quad \text{and} \quad \beta = \tilde{\beta} \ . \tag{2.59}$$

In addition, the transformation

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = L\begin{pmatrix} ct \\ x \end{pmatrix} \quad \text{with} \quad L \equiv \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \tag{2.60}$$

has to be unitary, that is,

$$1 = \det L = \gamma\tilde{\gamma} - \gamma\tilde{\gamma}\beta\tilde{\beta} = \gamma^2(1-\beta^2) \ , \tag{2.61}$$

which allows to relate the parameters γ and β by,

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \ . \tag{2.62}$$

Finally and obviously, we expect to recover the Galilei transform at low velocities,

$$ct = \gamma(ct' + \beta x') \to ct \quad \text{and} \quad x = \gamma(x' + \beta ct') \to x + ut \ . \tag{2.63}$$

That is, the limit is obtained by $\gamma \to 1$ and $\gamma\beta c \to u$, such that,

$$\beta = \frac{u}{c} \ . \tag{2.64}$$

such that the *Lorentz transform* from one inertial system S to another system S' is,

$$\begin{aligned} t' &= \gamma\left(t - \tfrac{u}{c^2}x\right) \quad \text{and} \quad x' = \gamma(x - ut) \quad \text{or} \\ t &= \gamma\left(t' + \tfrac{u}{c^2}x'\right) \quad \text{and} \quad x = \gamma(x' + ut') \ . \end{aligned} \tag{2.65}$$

2.2.4 Relativistic Doppler effect

We have seen at the example of sonic waves, that the magnitude of the Doppler effect depends on who moves with respect to the medium, whether it is the source or the listener. Electromagnetic waves, however, propagate in empty space, hence there is no material medium or wind. According to Einstein's theory of relativity, there is no absolute motion and the propagation velocity of light is the same for all inertial systems. Therefore, the theory of the sonic Doppler effect can not apply to electromagnetic waves. To deal with the Doppler effect of light, we need to talk a little about time dilation.

[4] Note that the calculus is dramatically simplified using the covariant formalism of 4-dimensional *space-time vectors* introduced by *Hermann Minkowski* and *Gregory Ricci-Curbastro*.

2.2.4.1 Dilation of time

We consider a clock flying through the lab S with the velocity v. The clock produces regular time intervals for which we measure in the lab the duration $t_2 - t_1$. The spatio-temporal points are Lorentz-transformed to the system S' in which the clock is at rest by,

$$\begin{pmatrix} ct'_j \\ z'_j \end{pmatrix} = \begin{pmatrix} \gamma ct_j - \gamma\beta z_j \\ -\gamma\beta ct_j + \gamma z_j \end{pmatrix} . \tag{2.66}$$

Hence,

$$\begin{aligned} t'_2 - t'_1 &= \gamma t_2 - \gamma\beta\frac{z_2}{c} - \gamma t_1 + \gamma\beta\frac{z_1}{c} \\ &= \gamma t_2 - \beta\left(\frac{z'}{c} + \gamma\beta t_2\right) - \gamma t_1 + \beta\left(\frac{z'}{c} + \gamma\beta t_1\right) = \gamma^{-1}(t_2 - t_1) . \end{aligned} \tag{2.67}$$

Consequently, in the lab the time interval seems longer than in the resting system.

Example 12 (*Doppler effect on a moving laser*): Coming back to the Doppler effect we now consider a light source flying through the lab S, for example, a laser operating at a frequency ω', which is well defined by an atomic transition of the active medium. A spectrometer installed in the same resting system S' as the laser will measure just this frequency. Now we ask ourselves, what frequency would a spectrometer installed in the lab measure. The classical response has been derived for a moving sound source,

$$\omega = \omega' - ku = \omega' - \frac{\omega}{c}u = \frac{\omega'}{1 + \frac{u}{c}} , \tag{2.68}$$

with $k = \omega/c$. But now, because of time dilation, we need to multiply by γ,

$$\omega = \frac{\gamma^{-1}\omega'}{1 + \frac{u}{c}} = \sqrt{\frac{1-\beta}{1+\beta}}\omega' \simeq \omega'\left(1 \pm \frac{u}{c} + \frac{u^2}{2c^2}\right) . \tag{2.69}$$

2.2.5 Exercises

2.2.5.1 Ex: Sonic Doppler effect

A speaker hanging from a wire of length $L = 1\,\mathrm{m}$ oscillates with a maximum angle of $\theta_m = 10°$ and emits a sound of $\nu = 440\,\mathrm{Hz}$.
a. What is the frequency of oscillation of the pendulum?
b. What is the energy $E_{cin} + E_{pot}$ of the oscillation?
c. What is the maximum oscillation speed?
d. What are the minimum and maximum frequencies of the sound perceived by a stationary receiver.

2.2.5.2 Ex: Sonic Doppler effect

Two identical speakers uniformly emit sound waves of $f = 680\,\mathrm{Hz}$. The audio power of each speaker is $P = 1\,\mathrm{mW}$. A point P is $r_1 = 2.0\,\mathrm{m}$ away from one device and

$r_2 = 3.0\,\mathrm{m}$ from the other.
a. Calculate the intensities I_1 and I_2 of the sound from each speaker separately at the P.
b. If the emission of the speakers were coherent and in phase, what would be the sound intensity in P?
c. If the emission of the speakers were coherent with a phase difference of 180°, what would be the sound intensity in P?
d. If the speaker output were incoherent, what would be the sound intensity in P?

2.2.5.3 Ex: Sonic Doppler effect

Suppose that a source of sound and a listener are both at rest, but the medium is moving relative to this frame. Will there be any variation in the frequency heard by the observer?

2.2.5.4 Ex: Sonic Doppler effect

Consider a source that emits waves of frequency f_{fnt} moving at velocity v_{fnt} on the x-axis. Consider an observer moving with velocity v_{obs} also on the x-axis. What will be the frequency perceived by the observer? Call the wave propagation velocity of c.

2.2.5.5 Ex: Sonic Doppler effect

Two trains travel on rails in opposite directions at velocities of the same magnitude. One of them is whistling. The whistle frequency perceived by a passenger on the other train ranges from 348 Hz when approaching to 259 Hz when moving away.
a. What is the velocity of the trains.
b. What is the frequency of the whistle.

2.2.5.6 Ex: Sonic Doppler effect

On a mountain road, while approaching a vertical wall which the road will surround, a driver is honking his horn. The echo from the wall interferes with the sound of the horn, producing 5 beats per second. Knowing that the frequency of the horn is 200 Hz, what is the speed of the car?

2.2.5.7 Ex: Sonic Doppler effect

A fixed sound source emits a sound of frequency ν_0. The sound is reflected by a fast approaching object (velocity u). The reflected echo returns to the source, where it interferes with the emitted waves giving rise to frequent beats $\Delta\nu$. Show that it is possible to determine the amplitude of the velocity of the moving object $|u|$ as a function of $\Delta\nu$, of ν_0, and of the speed of sound c.

2.2.5.8 Ex: Sonic Doppler effect

Two cars (1 and 2) drive in opposite directions on a road, with velocities of amplitudes v_1 and v_2. Car 1 travels against the wind, whose velocity is V. At sight of car 2

the driver of car 1 presses his horn, whose frequency is ν_0. The speed of sound in motionless air is c. What is the frequency ν of the horn sound perceived by the driver of car 2? What is the frequency ν' heard by the driver of a car 3 traveling in the same direction as car 1 and at the same speed?

2.2.5.9 Ex: Sonic Doppler effect

A physicist is molested by a fly orbiting his head. Since he is also a musician, he realizes that the sound of the buzz varies by one pitch. Calculate the speed of the fly.

2.2.5.10 Ex: Doppler effect

a. In a storm with wind velocity v a speaker well attached to the ground makes a sound of frequency f_0. How do you calculate the frequency recorded by a microphone taken by the wind and driven away from the speaker at the speed u?
b. Verify your answer in (a) by comparing the three cases (i) $u = 0$, (ii) $u = v$, and (iii) $v = 0$ with the cases of a moving source or receiver.

2.2.5.11 Ex: Sonic Doppler effect

A citizen of São Carlos is molested by a Tucano airplane operated by the Academia das Forças Aéreas de Pirassununga. He notices that while the airplane realizes looping on top of his head, the emitted sounds varies by up to an octave. Estimate the airplane's velocity.

2.3 Interference

The superposition of two counterpropagating waves can generate a *standing wave*. In these waves the oscillation amplitude depends on the position, but there is no energy transport.

2.3.1 Standing waves

We consider two waves $Y_\pm(x,t) = A\cos(kx \mp \omega t + \phi)$ propagating in opposite directions. In the case of a string this situation can be realized, e.g. by exciting a wave $Y_-(x,t)$ propagating in $-x$ direction, reflecting it subsequently at the end of the string ($x = 0$), and letting the wave $Y_+(x,t)$ propagate back in x direction,

$$Y(x,t) = Y_-(x,t) + Y_-(x,t) = A\cos(kx + \omega t) \pm A\cos(kx - \omega t) \ . \tag{2.70}$$

The sign of the reflected wave depends on how the end of the string is attached. If the end is fixed, the reflected wave inverts its amplitude. If it is free to move, the amplitude remains unaltered.

Let L be the length of the rope. The boundary conditions can be formulated as follows: When one end is clamped, the oscillation amplitude must be zero at this end,

$$Y(0,t) = 0 \qquad \text{or} \qquad Y(L,t) = 0 \ . \tag{2.71}$$

When one end is loose, the amplitude of oscillation must be maximum,

$$Y(0,t) = A \qquad \text{or} \qquad Y(L,t) = A \ . \tag{2.72}$$

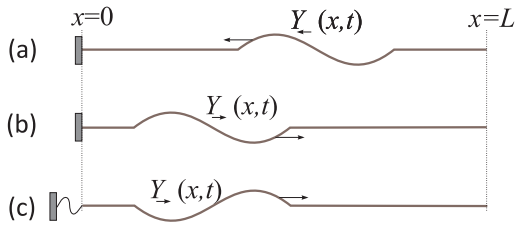

Figure 2.11: Superposition of a left-bound wave (a) with a wave reflected at a clamped end (b) or a loose end (c).

2.3.1.1 Rope with two ends fastened

In case that the two ends of the rope are clamped, we can simplify the superposition (2.70),

$$Y(x,t) = A\cos(kx + \omega t) - A\cos(kx - \omega t) = 2\sin kx \sin \omega t \ . \tag{2.73}$$

The boundary condition, $Y(L,t) = 0$, requires,

$$kL = \frac{2\pi L}{\lambda} = n\pi \ , \tag{2.74}$$

for a natural number n. This means that for a given length L and a given propagation velocity v, we can only excite oscillations satisfying,

$$\lambda = \frac{2L}{n} \qquad \text{and} \qquad \nu = \frac{v}{\lambda} = n\frac{v}{2L} \ . \tag{2.75}$$

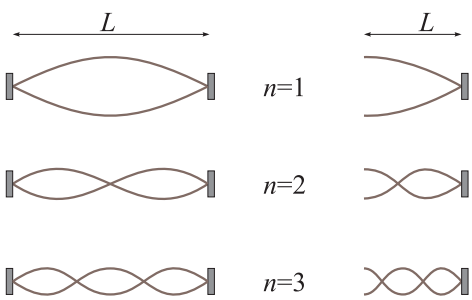

Figure 2.12: Vibration modes of a string fot (left) both ends tight up and (right) for one loose end.

2.3.1.2 Rope with one free end

In case that the end of the string at $x = 0$ is loose, we can simplify the superposition (2.70),

$$Y(x,t) = A\cos(kx + \omega t) + A\cos(kx - \omega t) = 2\cos kx \cos \omega t \ . \tag{2.76}$$

The boundary condition, $Y(L,t) = 0$, requires,

$$\phi = \tfrac{\pi}{2} \qquad \text{and} \qquad kL = \frac{2\pi L}{\lambda} = \left(n - \tfrac{1}{2}\right)\pi \ , \tag{2.77}$$

for a natural number n. This means that for a given length L and a given propagation velocity v, we can only excite oscillations satisfying,

$$\lambda = \frac{2L}{n - \frac{1}{2}} \qquad \text{and} \qquad \nu = \frac{v}{\lambda} = \left(n - \tfrac{1}{2}\right)\frac{v}{2L} \ . \tag{2.78}$$

Example 13 (*Stationary sound wave*):

- Exciting a stationary sound wave in a bottle.
- Exciting a standing sound wave on a guitar string.

2.3.2 Interferometry

2.3.2.1 Phase matching of two laser beams

When phase-matching two plane waves $\mathcal{E}_1 = Ae^{\imath\omega_1 t}$ and $\mathcal{E}_2 = Ae^{\imath\omega_2 t}$ on a photodiode, such that their wavevectors are parallel, the photodiode generates a *beat signal*,

$$I = |E_1 + E_2|^2 = AB[2 + 2\cos(\omega_1 - i\omega_2)t] \ . \tag{2.79}$$

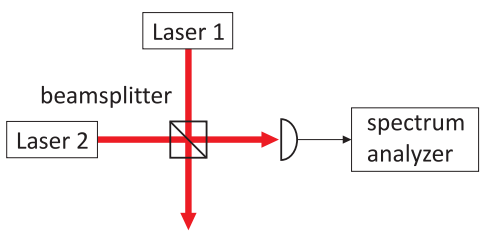

Figure 2.13: Principle of a beat frequency measurement.

In order to get a high signal contrast, a good phase-matching is important. It is particularly important to adjust the wavevectors to be absolutely parallel. In practice, however, this can be tricky, as the laser beams are frequently not plane waves, but have a finite diameter and radius of curvature.

Example 14 (*Laser interferometry*):

- Construct Michelson and Mach-Zehnder laser interferometers with one mirror mounted on a piezo. Show interference rings.

2.3.3 Diffraction

According to the Huygens principle, each point P_z within a slit emits a spherical wave reaching a given point P_k of the screen with a phase lag corresponding to the distance, as shown in Fig. 2.14,

$$r_{12} = \overline{P_\eta P_y} = \sqrt{(y-\eta)^2 + z^2} \ . \tag{2.80}$$

Thus, the phase difference between this ray and a ray coming out of the origin (which we place somewhere on the optical axis) is,

$$\phi = k\Delta r_{12} = k(\sqrt{(y-\eta)^2+z^2} - \sqrt{y^2+z^2}) \simeq -\frac{ky\eta}{\sqrt{y^2+z^2}} \simeq -\frac{ky\eta}{z} \equiv q\eta \ , \tag{2.81}$$

with $q = k\sin\alpha = ky/z$. If $A(\eta)$ is the amplitude of the excitation at the point η of the slit, then $B(y) = \frac{1}{z}e^{\imath\phi}$ is the amplitude at point y of the screen. Adding the contributions of all points,

$$B(q) = \sum_z e^{\imath\phi(y,z)} \to \int A(\eta)e^{\imath q\eta}d\eta \ . \tag{2.82}$$

We see that the amplitude distribution on the screen $B(y)$ is nothing more than the Fourier transform of the amplitude distribution $A(\eta)$ within the slit, regardless of the shape of the slit.

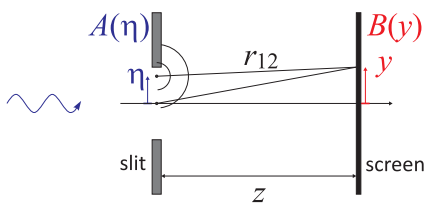

Figure 2.14: Fraunhofer diffraction at the slit.

The theory can be extended to 2D and 3D geometries, for example, a distribution of point-like scatterers within a given volume.

2.3.3.1 Single slit

As an example, we calculate the interference pattern behind a single slit. The Fourier transform of $A(\eta) = \chi_{[-d/2,d/2]}$ is,

$$B(q) = \int_{-d/2}^{d/2} e^{\imath q\eta}d\eta = \left.\frac{e^{\imath q\eta}}{\imath q}\right|_{-d/2}^{d/2} = d\frac{\sin\frac{1}{2}qd}{\frac{1}{2}qd} \ . \tag{2.83}$$

The intensity is $I(q) = c\varepsilon_0|B(q)|^2$.

2.3.3.2 Diffraction grating

We now calculate the interference pattern behind a diffraction grating with $N = 1000$ infinitely thin slits aligned within one millimeter. The Fourier transform of $A(\eta) =$

$\sum_{n=1}^{N} \chi_{[(n-1)d,(n-1)d+\Delta d]}$ is,

$$B(q) = \sum_{n=1}^{N} \int_{(n-1)d}^{(n-1)d+\Delta d} e^{\imath q\eta} d\eta = \frac{e^{\imath q\Delta d} - 1}{\imath q} \sum_{n=1}^{N} e^{\imath(n-1)qd} \tag{2.84}$$

$$\simeq \Delta d \sum_{n=0}^{N} e^{\imath nqd} = \Delta d \frac{1 - e^{\imath Nqd}}{1 - e^{\imath qd}} \ ,$$

where we approximated for $q\Delta d \ll 1$. For $N \to \infty$ we can approximate further,

$$B(q) = \frac{\Delta d}{1 - e^{\imath qd}} \ . \tag{2.85}$$

This is the *Airy function*, which is zero everywhere except at points where $qd = 2n\pi$. The intensity is,

$$I(q) = c\varepsilon_0 |B(q)|^2 = c\varepsilon_0 \frac{\Delta d^2}{2 - 2\cos qd} = c\varepsilon_0 \left(\frac{\Delta d}{2 \sin \frac{qd}{2}} \right)^2 \ . \tag{2.86}$$

The grating constant is $d = 0.001\,\mathrm{mm}$. The resulting pattern can be interpreted as

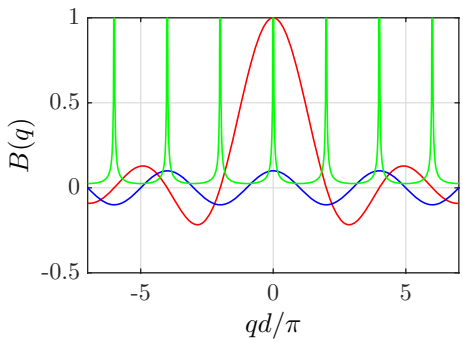

Figure 2.15: (code) Intensity distribution behind a diffraction grating for a single slit (red), a double slit (blue), and an infinite diffraction grating (green).

arising from a regular chain of antennas emitting synchronously. With a large number of point antennas, the chain emits in very well-defined directions. In addition, the direction can be controlled by arranging for a well-defined phase shift between the fields driving neighboring antennas.

2.3.4 Plane and spherical waves

In three dimensions the wave equation takes the form,

$$0 = \Box E \equiv \left(\frac{1}{c^2} \frac{\partial}{\partial t} - \nabla^2 \right) E \ . \tag{2.87}$$

In Cartesian coordinates, this gives,

$$0 = \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) E \ . \tag{2.88}$$

Plane waves, that is, waves described by the function,

$$Y(\vec{r},t) = Y_0 \sin(\vec{k}\cdot\vec{r} - \omega t) \ , \tag{2.89}$$

satisfy the wave equation if,

$$0 = -\frac{\omega^2}{c^2} + k_x^2 + k_y^2 + k_z^2 = -\frac{\omega}{c^2} + \vec{k}^2 \ . \tag{2.90}$$

2.3.4.1 Spherical waves

Spherical waves, that is, waves described by the function,

$$Y(\mathbf{r},t) = f(r)\sin(kr - \omega t) \ , \tag{2.91}$$

also satisfy the wave equation, provided the function $f(r)$ satisfies certain conditions. To find these conditions we use the representation of the *Laplace operator* in spherical coordinates,

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \ , \tag{2.92}$$

and insert the ansatz for $Y(\mathbf{r},t)$ into the wave equation. We have on one hand,

$$\frac{1}{c^2}\frac{d^2}{dt^2}(f\sin) = -\frac{\omega^2}{c^2} f\sin \ . \tag{2.93}$$

On the other hand,

$$\begin{aligned}\frac{1}{r}\frac{d^2}{dr^2}(rf\sin) &= \frac{1}{r}\frac{d}{dr}[f\sin + rf'\sin + krf\cos] \\ &= f''\sin + \frac{2f'}{r}\sin - k^2 f\sin + \frac{2k}{r}f\cos + 2kf'\cos \ ,\end{aligned} \tag{2.94}$$

such that,

$$0 = \Box f\sin = -\left(f'' + \frac{2f'}{r}\right)\sin - 2k\left(f' + \frac{f}{r}\right)\cos \ . \tag{2.95}$$

Thus the function f must satisfy the radial differential equation,

$$rf' + f = 0 \ . \tag{2.96}$$

This equation can be easily solved with the result $f(r) = r^{-1}$.

2.3.5 Formation of light beams

We consider monochromatic waves with frequency ω. Other waveforms can be synthesized by superpositions of waves with different frequencies. We also restrict to scalar waves. In fact, electromagnetic light fields are vectorial, however, close to the axis of an optical beam the fields are practically uniformly polarized, and representing

the amplitude of the field by a scalar wave is an excellent approximation. The field amplitude $\psi(\mathbf{r}, t)$ is governed by the following scalar wave equation,

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} \ . \tag{2.97}$$

We let ψ be of the form,

$$\psi(\mathbf{r}, t) = A(\mathbf{r})e^{\imath[\phi(\mathbf{r})-\omega t]} \ , \tag{2.98}$$

where A and ϕ are real functions of space. A is the *amplitude*, and the exponent is called the *phase* of the wave. In this form, it is implied that abrupt spatial or temporal variations are contained in the phase. The surface obtained by fixing the phase equal to a constant,

$$\phi(\mathbf{r}) - \omega t = const \tag{2.99}$$

is called *wave front* or *phase front*. The fast motion associated with a wave can be followed through the propagation of a particular wavefront. The interference between two waves is formed by the fronts of the two waves. The speed at which a particular wavefront is moving is called *phase velocity*. Suppose we follow a particular wavefront at the moment t: At time $t + \Delta t$, the phase front will have moved to another surface. A point $\mathbf{r}$ on the original surface will have moved to another point $\mathbf{r} + \Delta\mathbf{r}$ [see Fig. 2.16(a)]:

$$\phi(\mathbf{r} + \Delta\mathbf{r}) - \omega(t + \Delta t) = \phi(\mathbf{r}) - \omega t = const \tag{2.100}$$

Expanding $\phi(\mathbf{r} + \Delta\mathbf{r}) \simeq \phi(\mathbf{r}) + \nabla\phi(\mathbf{r})\Delta\mathbf{r}$, we obtain,

$$\nabla\phi(\mathbf{r})\Delta\mathbf{r} = -\omega t \ . \tag{2.101}$$

$\nabla\phi(\mathbf{r})$ is orthogonal to the phase front and is called the *wavevector*. $\Delta\mathbf{r}$ is smallest in the direction $\nabla\phi$, and the wavefront propagates with the velocity,

$$\frac{|\Delta\mathbf{r}|}{\Delta t} = \frac{\omega}{|\nabla\phi(\mathbf{r})|} \ . \tag{2.102}$$

which is the *phase velocity*. The phase velocity can vary from point to point in space.

Example 15 (*Phase velocity of a superposition of two plane waves*): The superposition of two plane waves with wavevectors $\mathbf{k}_{1,2} = k\hat{\mathbf{e}}_z \cos\theta \pm k\hat{\mathbf{e}}_x \sin\theta$ is described by,

$$\psi(\mathbf{r}, t) = A_0 e^{\imath(\mathbf{k}_1\cdot\mathbf{r}-\omega t)} + A_0 e^{\imath(\mathbf{k}_2\cdot\mathbf{r}-\omega t)} = 2A_0\cos(kx\sin\theta)e^{\imath(kz\cos\theta-\omega t)} \ . \tag{2.103}$$

The phase front of this wave is a plane with normal vectors pointing along the z-axis, as illustrated in Fig. 2.16(b), and the phase velocity is now $\omega/k\cos\theta = c/\cos\theta > c$.

2.3.5.1 Beam formation by superposition of plane waves

Plane waves extend throughout the space and are uniform in transverse direction, whereas an *optical beam* is confined in transverse direction. However, as we saw in the last example, by superposing two plane waves, a resulting wave can be obtained

which varies sinusoidally in transverse direction. By extrapolating this concept to superpositions of many plane waves, it is possible to construct by interference arbitrary transverse amplitude distributions. The propagation of a confined wave is the essence of *diffraction theory*. A particular case is the Gaussian beam. For mathematical simplicity and ease of visualization let us restrict ourselves to waves in two dimensions in the x-z plane. Only in the final phase will we present the complete results for three-dimensional Gaussian beams.

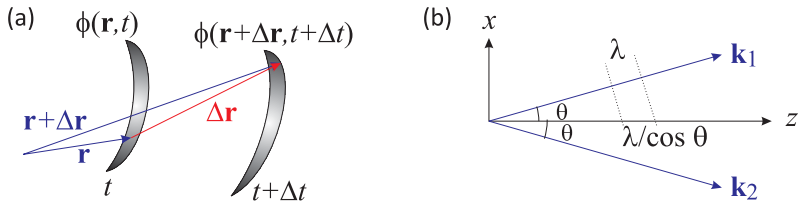

Figure 2.16: Superposition of two plane waves. The phase velocity along the direction z is higher than c, the speed of light, because in one period, the wavefront of each partial wave propagates over a distance λ, but along the z axis over a distance of $\lambda/\cos\theta$.

Before going into detailed calculations, we consider the last example again. The transverse standing wave resulting from the superposition of two plane waves, each one propagating at an angle θ with respect to the z-axis has a spatial frequency $k\sin\theta \simeq k\theta$ for small θ. We now come to a very important property of wave diffraction. Suppose that, in order to confine the wave in transverse direction, we continue adding plane waves, each one propagating at a different small angle θ, so that the amplitude adds constructively within the range $|x| < \Delta x$ and destructively out of it. By the uncertainty principle that results from the Fourier analysis and applies to this case,

$$\Delta(k\theta)\Delta x \gtrsim 1 \ . \tag{2.104}$$

That is, to confine a beam inside a width of Δx, it requires a distribution of plane waves in an angular spreading of at least $\lambda/2\pi\Delta x$. The angular spreading means that the beam will eventually diverge with an angle $\Delta\theta$.

2.3.5.2 Fresnel integrals and beam propagation

Let us now superpose plane waves in a way to form a beam. Each partial wave propagates under some angle θ with respect to the z-axis and has an amplitude $A(\theta)d\theta$, so that the resulting wave (omitting the harmonic temporal variation) is,

$$\psi(x,z) = \int d\theta A(\theta) e^{\imath kx\sin\theta + \imath kz\cos\theta} \ . \tag{2.105}$$

In the so-called *paraxial approximation*, $A(\theta)$ is significant only within a small angular interval close to zero. This means that, according to Eq. (2.104), the transverse dimension of the beam is large in comparison to the wavelength. Expanding the trigonometric functions up the order θ^2,

$$\psi(x,z) \simeq \int d\theta A(\theta) e^{\imath kx\theta + \imath kz(1-\theta^2/2)} = e^{\imath kz} \int d\theta A(\theta) e^{\imath kx\theta - ikz\theta^2/2} \ . \tag{2.106}$$

This wave can be considered as a plane wave, $e^{\imath kz}$, modulated by the integral of (2.106). The expression (2.106) completely describes the propagation of the wave, provided that the wave is known at some point, say $z = 0$. In fact, at $z = 0$, the expression (2.106) for $\psi_0(x) \equiv \psi(x, 0)$ is a Fourier transform, whose inverse yields the angular distribution,

$$A(\theta) = \frac{k}{2\pi} \int d\xi \psi_0(\xi) e^{-\imath k\xi\theta} \ . \tag{2.107}$$

Substitution of $A(\theta)$ back into Eq. (2.106) gives,

$$\psi_z(x) \equiv \psi(x, z) = \frac{k}{2\pi} e^{\imath kz} \int d\theta \int d\xi \psi_0(\xi) e^{\imath(k\theta x - k\theta\xi - kz\theta^2/2)} \ . \tag{2.108}$$

From here on, in order to emphasize the different roles played by the transverse coordinates x and y, we will label the axial position z as an index to the wave function.

We can first integrate over θ via a quadratic extension of the exponent. The result,

$$\frac{k}{2\pi} e^{\imath kz} \int d\theta e^{\imath(k\theta x - k\theta\xi - kz\theta^2/2)} = \sqrt{\frac{k}{2\pi\imath z}} e^{\imath k(z + (x-\xi)^2/2z} \equiv h_z(x - \xi) \ , \tag{2.109}$$

gives us the field at the position z as an integral over ξ of the field in $z = 0$, $\psi_0(\xi)$. The expression (2.109) is called *impulse response*, *kernel*, *propagator*, or *Green's function*, depending on the context. Carry out the integral (2.109) in Exc. 2.3.6.17.

The kernel has very simple physical interpretations: It is the field at point (x, z) generated by a point source with unitary amplitude located in $(\xi, 0)$. In the same time, it is a (two-dimensional) spherical wave in a paraxial form. To see this, we write the field of a two-dimensional spherical wave (i.e. a circular wave) with its center in $(\xi, 0)$ as,

$$\sqrt{\frac{1}{r}} e^{\imath kr} \ , \tag{2.110}$$

where $r = \sqrt{(x - \xi)^2 + z^2}$. (Instead of $1/r$ as in three dimensions, the amplitude decreases as $\sqrt{1/r}$ in two dimensions.) Near the z-axis, we approximate $r \simeq z + (x - \xi)^2/2z$, and the spherical wave becomes,

$$\sqrt{\frac{1}{z}} e^{\imath k[z + (x-\xi)^2/2z]} \ , \tag{2.111}$$

which is the same expression as $h_z(x - \xi)$ in Eq. (2.109). Note that the quadratic term in $x - \xi$ can become considerable in comparison with the wavelength. Eq. (2.106) now becomes,

$$\boxed{\psi_z(x) = \int h_z(x - \xi)\psi_0(\xi) d\xi = \sqrt{\frac{k}{2\pi\imath z}} e^{\imath kz} \int e^{\imath k(x-\xi)^2/2z} \psi_0(\xi) d\xi} \ . \tag{2.112}$$

We will call this integral the *Fresnel integral*. It is the mathematical expression of the *Huygens principle*: The field in (x, z) is the sum of all spherical waves centered on all previous points $(\xi, 0)$ weighed with the respective field amplitude $\psi_0(\xi)$ [1].

The expressions (2.106) and (2.112) represent two equivalent ways to calculate wave propagation. Eq. (2.106) calculates the wave from the angular distribution of

its plane wave components. When the angular distribution is of Hermite-Gaussian type, a Gaussian beam results. In contrast, Eq. (2.112) computes the wave at a point z from the field at an initial point $z = 0$. This is the traditional theory of Fresnel diffraction. Here, also a Gaussian beam results when ψ_0 is Hermite-Gaussian.

To deepen our understanding of beam propagation let us introduce the important concept of *near field* and *far field*. By 'near field' we mean a distance z sufficiently small to be allowed to neglect the quadratic term in the exponent of Eq. (2.106),

$$k\theta^2 z \ll 1 \ . \tag{2.113}$$

Then the near field, in zero-order approximation, is precisely the field at $z = 0$ multiplied with propagation phase factor $e^{\imath kz}$,

$$\psi_z(x) \simeq e^{\imath kz} \int d\theta A(\theta) e^{\imath kz\theta} = e^{\imath kz}\psi_0(x) \ , \tag{2.114}$$

where the second equation follows from Eq. (2.107). Let us now examine the first-order correction and define 'near' more precisely.

The question is, what is the maximum angle of θ allowed in (2.113)? It is not $\pi/2$, but rather, it is the range of angles over which $A(\theta)$ is significantly different from zero. This angular range $\Delta\theta$ is related to the range of transverse distance Δx via the Fourier transform (2.104), so that,

$$\frac{\pi \Delta x^2}{\lambda} \gg z/2 \ . \tag{2.115}$$

The quantity on the left side, called the *Rayleigh range*, is the demarcation between the near and far field regimes. A simple physical interpretation for this quantity will be given below.

Let us now investigate the 'far field' regime of large z having a closer look at Eq. (2.112). When ψ_0 is confined to Δx, and if z is sufficiently large for the quadratic factor to be,

$$k\xi^2/2z \ll 1 \ , \tag{2.116}$$

or

$$\frac{\pi \Delta x^2}{\lambda} \ll z \ , \tag{2.117}$$

then it can be ignored, and the integral becomes,

$$\psi(x,z) \simeq \sqrt{\frac{k}{\imath 2\pi z}} e^{\imath k(z+x^2/2z)} \int e^{-\imath kx\xi/z} \psi_0(\xi) d\xi \ . \tag{2.118}$$

We see that the amplitude of the far field is given by the amplitude of the Fourier transform of the field at $z = 0$ except for a quadratic phase factor $kx^2/(2z)$ [5]

Let us go back to the near field and calculate the first-order correction. For small $z = \Delta z$, we can expand the exponent in equation (2.106),

$$\begin{aligned} \psi_z(x) &\simeq e^{\imath k\Delta z} \int d\theta A(\theta) \left(1 - \imath k \frac{\theta^2}{2} \Delta z\right) e^{\imath k\theta x} \\ &= e^{\imath k\Delta z}\psi_0(x) - e^{\imath k\Delta z} \frac{\imath k \Delta z}{2} \int d\theta A(\theta)\theta^2 e^{\imath k\theta x} \ . \end{aligned} \tag{2.119}$$

[5] In fact, the phase factor can be circumvented by choosing z equal to a focal length f of a lens.

The last integral is,

$$\int d\theta A(\theta)\theta^2 e^{\imath k\theta x} = -\frac{1}{k^2}\frac{\partial^2}{\partial x^2}\int d\theta A(\theta)e^{\imath k\theta x} = -\frac{1}{k^2}\frac{\partial^2\psi_0(x)}{\partial x^2} \ , \tag{2.120}$$

Such that close to $z = 0$, we get,

$$\psi_z(x) \simeq e^{\imath k\Delta z}\left[\psi(x,0) + \frac{\imath\Delta z}{2k}\frac{\partial^2\psi(x,0)}{\partial x^2}\right] \ . \tag{2.121}$$

Note that the first-order correction is in quadrature with the zero-order term (2.114) (if ψ_0 is real), which means that the correction is in the phase, not in the amplitude. The second derivative can be seen as a diffusion operator [6], and it is this phase diffusion, which is the cause of phenomenon of *diffraction*.

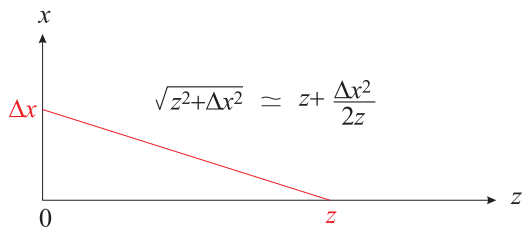

Figure 2.17: Illustration of the Rayleigh range: The distance from $(0,0)$ to $(0,z)$ is z. The distance from $(0,\Delta x)$ to $(0,z)$ is approximately $z + \Delta x^2/(2z)$. The difference is $\Delta x^2/(2z)$. Thus, a wave coming from $(0,0)$ and a wave coming from $(0,\Delta x)$ will acquire a phase difference of $k\Delta x^2/(2z)$ when they reach $(0,z)$. The phase difference is equal to 1 when z equals the Rayleigh range. The phase difference is insignificant in the far field, but significant in the near field.

We can generalize a little more: Suppose we write ψ as a plane wave $e^{\imath kz}$ modulated by a function with slow variation $u(x,z)$,

$$\psi_z(x) \equiv u_z(x)e^{\imath kz} \ , \tag{2.122}$$

then

$$u_{\Delta z}(x) - u_0(x) = \frac{\imath\Delta z}{2k}\frac{\partial^2 u_0(x)}{\partial x^2} \ . \tag{2.123}$$

We derived this relation for a particular point on the z-axis, $z = 0$. However, there is no particular need to choose this point, and the relationship applies to any z. Thus, letting $\Delta z \to 0$, we get,

$$2ik\frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial x^2} = 0 \ . \tag{2.124}$$

This equation is called *paraxial wave equation*. It is an approximate form of the scalar wave equation and has the same form as the Schrödinger equation for a free particle. The equation can be generalized to three dimensions by a similar derivation:

$$\boxed{2\imath k\frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \ . \tag{2.125}$$

[6] This is because the second derivative of a Gaussian function is negative in the center and positive in the wings, so that when added to the original function, the distribution is reduced in the center and increased in the wings.

The *Fresnel integral is the solution of the paraxial wave equation* with a boundary condition for ψ at $z = 0$. We will show in Sec. ?? that a three-dimensional wave can be constructed from two-dimensional waves [2]. The resulting Fresnel integral in three dimensions is,

$$\boxed{\psi_z(x,y) = \frac{e^{\imath kz}}{\imath\lambda z}\int e^{\imath k(x-\xi)^2/2z} e^{\imath k(y-\eta)^2/2z}\psi_0(\xi,\eta)d\xi d\eta}\ , \tag{2.126}$$

where $\psi_0(x,y)$ is the distribution of the field amplitude at $z = 0$. Note that, as required by energy conservation, in three dimensions the field decays like $1/z$ and not like $\sqrt{1/z}$, as it does in two dimensions. Note also that the pulse response in three dimensions is essentially the product of two two-dimensional pulse responses.

2.3.5.3 Application of Fresnel diffraction theory

The Fresnel diffraction integral, Eq. (2.112), can be applied in various situations illustrating its use and the difference between wave optics and geometric optics. Examples are the diffraction through a slit, the pin-hole camera, the focusing of a thin lens, etc. [2].

Near-field diffraction (also called *Fresnel diffraction*) and far-field diffraction (also called *Fraunhofer diffraction*) are often distinguished by a quantity called the *Fresnel number*,

$$F \equiv \frac{a^2}{z\lambda}\ , \tag{2.127}$$

where a is the size of the beam (or *aperture*). The near field zone is defined by $F \gtrsim 1$, whereas in the far field zone, $F \ll 1$. For a Gaussian beam, letting $a = \sqrt{\pi}w_0$, we recover the Rayleigh length condition for Fresnel diffraction $z \lesssim z_R$, respectively Fraunhofer diffraction, $z \gg z_R$.

2.3.6 Exercises

2.3.6.1 Ex: Waves on a rope

A string with linear mass density μ is attached at two points distant by $L = 1\,\mathrm{m}$. A mass of $m = 1\,\mathrm{kg}$ is attached to one end of the string that goes over a pulley, as shown in the figure. Excited by a vibrating pin with frequency $f = 1\,\mathrm{kHz}$ the string performs transverse vibrations with the wavelength $\lambda = 2L$.
a. Calculate the sound velocity.
b. Now the mass is replaced by a mass $m' = 4m$. Calculate the new sound velocity.
c. Assuming the sound velocity, how often should the pin excite the string to observe the third oscillation mode (three anti-nodes)?

2.3.6.2 Ex: Optical cavity

Optical cavities consist of two light reflecting mirrors. Standing light waves must satisfy the condition that the electric and magnetic fields vanish on the mirror surfaces. What is the frequency difference between two consecutive modes of a of length $L = 10\,\mathrm{cm}$?

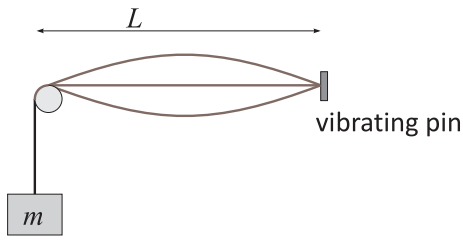

Figure 2.18: Waves on a rope.

2.3.6.3 Ex: Waves on a rope

A string vibrates according to the equation $y(x,t) = 15\,\sin\frac{\pi x}{4\,\mathrm{cm}}\cos(30\,\mathrm{s}^{-1}\,\pi t)$.
a. What is the velocity of a string element at the position $x = 2\,\mathrm{cm}$ at the instant of time $t = 2\,\mathrm{s}$?
b. What is the propagation speed of this wave?

2.3.6.4 Ex: Violin

The length of a violin string is $L = 50\,\mathrm{cm}$, and its mass is $m = 2.0\,\mathrm{g}$. When it is attached at the ends, the string can emit the a'-pitch ('la') corresponding to 440 Hz. Where should a finger be placed so that the emitted sound is the c"-pitch ('do') at 528 Hz?

2.3.6.5 Ex: Sound waves

The air column inside a closed tube, filled with a gas whose characteristic sound velocity is v_s, is excited by a speaker vibrating at the frequency f. Gradually increasing the frequency of the speaker one observes that the tube emits a sound at $f = 440\,\mathrm{Hz}$ and the next time at 660 Hz.
a. What is the length of the tube?
b. What is the speed of sound?

2.3.6.6 Ex: Sound in a bottle

An experimenter blows into a bottle partially filled with water producing a sound of 1000 Hz. After drinking some of the water until the level decreased by 5 cm he is able to produce a sound at 630 Hz. Determine the possible values for the speed of sound knowing that the vibration of the air column inside the bottle should have a node at the end which is in contact with water and an anti-node at the mouth of the bottle. Comparing the result to the known value for the speed of sound in air, what is the excited vibration mode?

2.3.6.7 Ex: Sonic waves in a tube

The figure shows a rod fixed at its center to a vibrator. A disc attached to the end of the rod penetrates a glass tube filled with a gas and where cork dust had been deposited. At the other end of the tube there is a movable piston. When producing

longitudinal vibrations at the rod, we note that for certain positions of the movable piston, the cork dust forms a pattern of node and anti-nodes. Knowing for one of the positions of the piston the distance d between the anti-nodes and the frequency f of the vibration, show that the speed of sound in the gas is $v = 2fd$. This is called Kundt's method for determining the speed of sound in a gas.

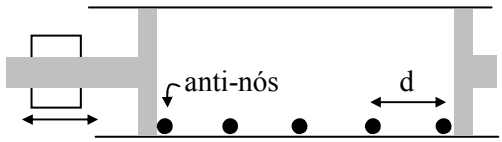

Figure 2.19: Sonic waves in a tube.

2.3.6.8 Ex: Sound filter

A tube can act as an acoustic filter discriminating various sound frequencies crossing it from its own frequencies. A car muffler is an application example.
a. Explain how this filter works.
b. Determine the 'cut-off' frequency below which sound is not transmitted.

2.3.6.9 Ex: Snell's law

Derive Snell's law from Huygens principle.

2.3.6.10 Ex: Surface gravitational waves, capillary waves

Dependence of the propagation velocity on the height of the water column.

2.3.6.11 Ex: Propagating standing wave

Consider two propagating waves $\mathcal{E}_{\pm}(x,t)$ with equal amplitudes and slightly different frequencies $\omega_{\pm}$ propagating in opposite directions along the x-axis.
a. Show that, approximating $k_+ \simeq k_-$, at each instant of time the interference pattern along the x-axis forms a standing wave.
b. Determine the group velocity of this wave.

2.3.6.12 Ex: Mach-Zehnder and Michelson-interferometer

Interferometers are devices that allow the comparison of distances via the propagation time of waves taking different paths. The interferometers outlined in the figures are based on beam splitters that divide and recombine a wave described by $I_n(x,t) = A_n \cos(kx - \omega t)$. Determine the amplitude of the signal at the position of the beam splitter recombining the waves as a function of a variation $\Delta x = 4\pi/k$ of the length of the second interferometer arm.

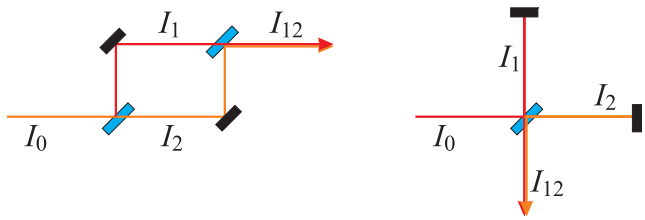

Figure 2.20: Mach-Zehnder and Michelson-interferometer.

2.3.6.13 Ex: Multiple interference in optical cavities

An optical beam splitter is a mirror with partial transmission and partial reflection,

$$\mathcal{E}_r(x,t) = \pm r\mathcal{E}_0(x,t) \quad , \quad \mathcal{E}_t(x,t) = t\mathcal{E}_0(x,t) \ .$$

The reflection signal depends on the direction of incidence, because reflection at a denser medium introduces a phase shift of π. Using this rules derive for a set of two mirrors r_1 and r_2 separated by a distance L the field $\mathcal{E}_{cav}$ between the mirrors as a function of the wave vector of the incident field $\mathcal{E}_{in}$. Also calculate the amplitudes of the transmitted and reflected light. Calculate the phase shifts between the transmitted (reflected) light and the incident light. Interpret the results.

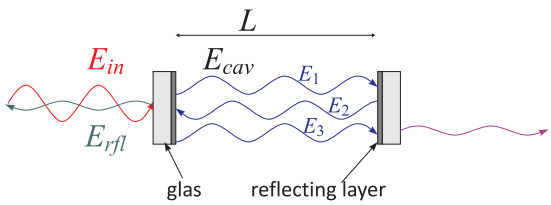

Figure 2.21: Optical cavity.

2.3.6.14 Ex: Double slit

Calculate the interference pattern behind a double slit.

2.3.6.15 Ex: Spherical waves

Show that spherical waves given by $Y(\mathbf{r},t) = \frac{Y_0}{kr}\sin(kr - \omega t)$ satisfy the 3D wave equation. Use Cartesian coordinates.

2.3.6.16 Ex: Interference in spherical waves

Two spherical waves are generated at positions $\mathbf{r}_\pm = \pm R\hat{\mathbf{e}}_z$. Determine surfaces of destructive interference for these waves.

2.3.6.17 Ex: Green's function

Calculate the integral Eq. (2.109).

2.4 Fourier analysis

Every periodic function $f(\xi) = f(\xi+2\pi)$ can be decomposed into a series of harmonic vibrations. This is the *Fourier theorem*,

$$\boxed{f(\xi) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\xi + b_n \sin n\xi)} \ . \tag{2.128}$$

To determine the coefficients, we calculate,

$$\int_0^{2\pi} f(\xi)d\xi = \int_0^{2\pi} \left[\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos m\xi + b_m \sin m\xi\right] d\xi = \pi a_0 \tag{2.129}$$

$$\int_0^{2\pi} f(\xi)\cos k\xi d\xi = \int_0^{2\pi} \left[\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos m\xi + b_m \sin m\xi\right] \cos n\xi d\xi = \pi a_n$$

$$\int_0^{2\pi} f(\xi)\sin k\xi d\xi = \int_0^{2\pi} \left[\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos m\xi + b_m \sin m\xi\right] \sin n\xi d\xi = \pi b_n \ ,$$

using the rules,

$$\int_0^{2\pi} \cos n\xi \cos m\xi d\xi = \int_0^{2\pi} \sin n\xi \sin m\xi d\xi = \pi\delta_{n,m} \quad \text{and} \quad \int_0^{2\pi} \cos n\xi \sin m\xi dx = 0 \ . \tag{2.130}$$

We can use these equations to calculate the Fourier expansion. To simplify the calculations, it is useful to consider the symmetry of the periodic function, since if $f(\xi) = f(-\xi)$, we can neglect all the coefficients b_n, and if $f(\xi) = -f(-\xi)$, we can neglect the coefficients b_n [7]

Example 16 (*Frequency spectrum and low-pass filter*):

- Show the spectrum of a rectangular signal on an oscilloscope and on an spectrum analyzer.
- Show the same spectrum filtered by a low pass filter.

.

[7] Alternatively we can write the theorem as,

$$f(\xi) = \sum_{n=-\infty}^{\infty} d_n e^{\imath n\xi} \ ,$$

determining the coefficients as,

$$\int_{-\pi}^{\pi} f(\xi)e^{-\imath k\xi}d\xi = \int_{-\pi}^{\pi} f(\xi)e^{-\imath k\xi}d\xi \sum_{n=-\infty}^{\infty} d_n e^{\imath n\xi}d\xi = 2\pi d_n \ ,$$

with,

$$2d_n = a_n - \imath b_n \quad \text{for} \quad n \geq 0 \quad \text{and} \quad 2d_n = a_{-n} + \imath b_{-n} \quad \text{for} \quad n < 0 \ .$$

2.4.1 Expansion of vibrations

Interpreting $\xi \equiv \omega t$ as time, we can apply the Fourier theorem (2.128) on temporal signals, $S(t) = f(\omega t)$, where ω is the angular frequency,

$$S(t) = f(\omega t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t) \ , \tag{2.131}$$

with,

$$a_0 = \frac{\omega}{\pi}\int_0^{2\pi/\omega} S(t)dt \quad \text{and} \quad a_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} S(t)\cos n\omega t dt \tag{2.132}$$
$$\text{and} \quad b_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} S(t)\sin n\omega t dt \ .$$

The representation of the coefficients a_n and b_n as functions of the number n is called *harmonic spectrum*. As we mentioned earlier, the spectrum of a sound is what determines the timbre. The total *harmonic distortion* is defined by,

$$k \equiv \frac{\sum_{n=2}^{\infty}(a_n + b_n)}{\sum_{n=1}^{\infty}(a_n + b_n)} \ . \tag{2.133}$$

Radiofrequency circuits such as HiFi amplifiers are characterized by their transmission fidelity, that is, the absence of harmonic distortion in the amplification of each harmonic coefficient.

2.4.1.1 Expansion of a triangular signal

We consider a *triangular signal* given by [8],

$$S(t) = \begin{cases} \omega t & \text{for} \quad 0 < \omega t < \frac{\pi}{2} \\ \pi - \omega t & \quad\quad \frac{\pi}{2} < \omega t < \pi \end{cases} \ . \tag{2.134}$$

We calculate the coefficients, $a_0 = 0$, because the signal is symmetric about the t-axis (it has no offset), and $a_n = 0$, because the signal has the symmetry $S(t) = -S(-t)$. Also,

$$b_n = \frac{2\omega}{\pi}\int_0^{\pi/2\omega} \omega t \sin n\omega t dt + \frac{2\omega}{\pi}\int_{\pi/2\omega}^{\pi/\omega}(\pi - \omega t)\sin n\omega t dt = \frac{4}{\pi}\frac{\sin\frac{1}{2}\pi n}{n^2} \ , \tag{2.135}$$

with the consequence,

$$S(t) = \frac{4}{\pi}\sum_{n=1,3,\ldots}\frac{(-1)^{(n-1)/2}}{n^2}\sin n\omega t \ . \tag{2.136}$$

[8] Note that the function $S(t) = \frac{\pi}{2} - \left(\frac{\pi}{2} - \omega t\right)\frac{\cos\omega t}{|\cos\omega t|}$, which describes the same triangular signal, it is easier to program in numerical softwares.

2.4.2 Theory of harmony

Non-linearities in oscillating systems can excite harmonic frequencies f_n, that is, multiples of the fundamental frequency $f_n = (n+1)f$. These are the components of the Fourier series.

All musical instruments produce harmonics. This is what makes the timbre of the instrument. When we play several notes together, we perceive the octave interval as pleasant. This is, because all the harmonics of a pitch and of its octave coincide.

- harmonic pitch, well-tempered chromatic scale, flat ♭, sharp ♯, ♮, musical clef, tuning fork $f_{a'} = 440\,\mathrm{Hz}$

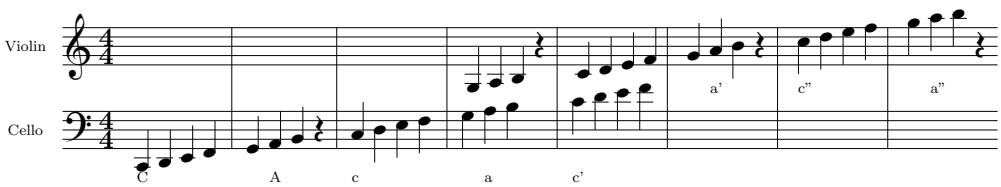

Figure 2.22: (code) Ladder of pitches over 3 octaves. Pitches can be generated in MATLAB. A sample program can be downloaded by clicking on the link.

In the *well-tempered* tonality the interval of a *octave* is divided into 12 intervals,

$$n \in [\text{a, a\#, h, c, c\#, d, d\#, e, f, f\#, g, g\#}] \ . \tag{2.137}$$

Defining *normal tuning* as,

$$f_a = 440\,\mathrm{Hz} \ , \tag{2.138}$$

the pitches correspond to the frequencies,

$$f_n = 2^{n/12} f_a \ . \tag{2.139}$$

For example, we calculate the frequency of the 'd',

$$f_d = 2^{-7/12} f_a = 391.9954\,\mathrm{Hz} \ . \tag{2.140}$$

Thus, all notes are logarithmically equidistant:

$$\mathrm{lb}\ f_{n+1} - \mathrm{lb}\ f_n = \mathrm{lb}\ (2^{(n+1)/12} f_{la}) - \mathrm{lb}\ (2^{n/12} f_{la}) = 1 \tag{2.141}$$

$$\frac{f_{n+1}}{f_n} = \frac{2^{(n+1)/12} f_{la}}{2^{n/12} f_{la}} = 2^{1/12} \ . \tag{2.142}$$

Why are there just 12 pitches? Several instruments have more than one resonator emitting sound, e.g. the violoncello has 4 strings, c, g, d', and a'. Each string is detuned by a *quint* from the next string, that is,

$$3f_c = 2f_g \quad \text{and} \quad 3f_g = 2f_{d'} \quad \text{and} \quad 3f_{d'} = 2f_{a'} \ . \tag{2.143}$$

Each string has its own series of harmonics. The timbre of the instrument appears more pleasant, when the harmonics of the various strings coincide. Let us now check, whether our definition of logarithmically equidistant pitches satisfies this condition,

$$3f_d = 3 \cdot 2^{-7/12} f_a = 2.0023 f_a \neq 2 f_a \ . \tag{2.144}$$

Thus, harmonic tuning is not perfect, but quite close to the well-tempered tuning. In Exc. 2.4.6.5 we show that, nevertheless, the discrepancy is able to produce nasty beat notes.

The guitar, which is tuned in *quart*s,

$$4f_c = 3f_{fa} \ , \tag{2.145}$$

has the same problem [9],

$$4f_c = 4 \cdot 2^{-5/12} f_f = 2.9966 f_f \neq 3 f_f \ . \tag{2.146}$$

Resolve the Excs. 2.4.6.6, 2.4.6.7 and 2.4.6.8.

2.4.3 Expansion of waves

Interpreting $\xi \equiv kx$ as position, we can apply the Fourier theorem (2.128) to standing waves, $Y(x) = f(kx)$, where $k = 2\pi/k$ is the wavevector.

2.4.4 Normal modes in continuous systems at the example of a string

We will now apply the Fourier expansion to calculate the normal modes of a vibrating string. Depending on which mode of oscillation is excited, the displacement of the string is given by,

$$Y_n(x,t) = (A_n \cos\omega_n t + B_n \sin\omega_n t) \sin\tfrac{\omega_n x}{c} \ , \tag{2.147}$$

where $\omega_n = n\pi c/l$ is the frequency of the *normal mode*. An arbitrary vibration can be decomposed as superpositions of these modes,

$$Y(x,t) = \sum_n Y_n(x,t) \ . \tag{2.148}$$

As an initial condition we assume that the string is at a position $Y(x,0) = Y_0(x)$ with the velocity $V(x,0) = V_0(x)$ at all points. Then,

$$\begin{aligned} Y_0(x) &= \sum_n Y_n(x,0) = \sum_n A_n \sin\tfrac{\omega_n x}{c} \\ \text{and} \qquad V_0(x) &= \sum_n \frac{d}{dt} Y_n(x,0) = \sum_n \omega_n B_n \sin\tfrac{\omega_n x}{c} \ . \end{aligned} \tag{2.149}$$

[9] Include Matlab sound examples here!

We find the amplitudes by calculating the integrals,

$$\frac{2}{l}\int_0^l Y_0(x)\sin\tfrac{\omega_n x}{c}dx = \frac{2}{l}\sum_m A_m\int_0^l \sin\tfrac{m\pi x}{l}\sin\tfrac{n\pi x}{l}dx = A_n \tag{2.150}$$
$$\frac{2}{l}\int_0^l V_0(x)\sin\tfrac{\omega_n x}{c}dx = \frac{2}{l}\int_0^l \sum_m \omega_n B_m \sin\tfrac{m\pi x}{l}\sin\tfrac{n\pi x}{l}dx = \omega_n B_n \ .$$

We now assume that the rope is initially excited by a triangular deformation, that is, we pull the rope in its middle up to a distance d and let go. That is, the initial conditions are given by,

$$V_0(x) = 0 \qquad \text{and} \qquad \frac{\pi}{2d}Y_0(x) = \begin{cases} \frac{\pi x}{l} & \text{for} \quad 0 < \frac{\pi x}{l} < \frac{\pi}{2} \\ \frac{\pi(l-x)}{l} & \quad\quad\ \ \frac{\pi}{2} < \frac{\pi x}{l} < \pi \end{cases} \ . \tag{2.151}$$

We can compare this function with the triangle function Eq. (2.134) and make the same Fourier expansion as in (2.136),

$$\frac{\pi}{2d}Y_0(x) = \frac{4}{\pi}\sum_{n=1,3,\ldots}\frac{(-1)^{(n-1)/2}}{n^2}\sin\tfrac{n\pi x}{l} \ . \tag{2.152}$$

Comparing this expansion with (2.149), we find $B_n = 0$ and,

$$\sum_m A_m \sin\tfrac{\omega_m x}{c} = Y_0(x) = \frac{2d}{\pi}\frac{4}{\pi}\sum_{n=1,3,\ldots}\frac{(-1)^{(n-1)/2}}{n^2}\sin\tfrac{n\pi x}{l} \ . \tag{2.153}$$

yielding for odd coefficients $m = 1, 3, ..,$

$$A_n = \frac{8d}{n^2\pi^2}(-1)^{(n-1)/2} \ . \tag{2.154}$$

Thus, the vibration of the string is completely described by,

$$Y(x,t) = \frac{8d}{\pi^2}\sum_{n=1,3,\ldots}\frac{(-1)^{(n-1)/2}}{n^2}\cos\omega_n t\sin\tfrac{\omega_n x}{c} \ . \tag{2.155}$$

The energy is the sum of the energies of all normal modes,

$$E = \sum_{n=1,3,\ldots}\frac{m}{4}\omega_n^2 A_n^2 = \sum_{n=1,3,\ldots}\tfrac{m}{4}\left(\frac{n\pi c}{l}\right)^2\left(\frac{8d}{n^2\pi^2}\right)^2 = \sum_{n=1,3,\ldots} m\frac{16d^2c^2}{n^2\pi^2 l^2} = \frac{2md^2c^2}{l^2} \ , \tag{2.156}$$

knowing $\sum_{n=1,3,\ldots}\frac{1}{n^2} = \frac{\pi^2}{8}$.

2.4.5 Waves in crystalline lattices

The sound may propagate in a crystalline lattice, for example a metal or a crystal, by means of longitudinal or transverse vibrations. To understand the propagation of longitudinal vibrations in a monoatomic lattice, we consider the model of a chain of

N masses coupled by springs. The treatment for transverse vibrations is analogous. As we have shown in previous sections, the movement of each mass is described by the differential equation,

$$\ddot{x}_n = \omega_0^2(x_n - x_{n-1}) + \omega_0^2(x_n - x_{n+1}) \ , \tag{2.157}$$

with $n = 1, .., N$. Making the ansatz $x_n = A_n e^{-\imath\omega t}$, we obtain the characteristic equation,

$$\omega^2 A_n = \omega_0^2(A_n - A_{n-1}) + \omega_0^2(A_n - A_{n+1}) \ . \tag{2.158}$$

When we hit one of the oscillators of a linear chain, we excite a wave that propagates along the chain. Therefore, it is reasonable to guess $A_n = Ae^{\imath nka}$ for the displacements of the oscillators, where $a \equiv x_{n+1} - x_n$ is the lattice constant. We obtain,

$$\omega^2 = \omega_0^2(1 - e^{-ika}) + \omega_0^2(1 - e^{\imath ka}) = 2\omega_0^2(1 - \cos ka) = 4\omega_0^2 \sin^2 \frac{ka}{2} \ . \tag{2.159}$$

The dispersion relation is shown in Fig. 2.23. Obviously, in the limit of long waves, $ka \ll 1$, the relation can be approximated by,

$$\omega = 2\omega_0 \left| \sin \frac{ka}{2} \right| \simeq \omega_0 ka \equiv ck \ , \tag{2.160}$$

where c is the propagation velocity of the wave. This relation is linear, thus reproducing the situation of acoustic waves.

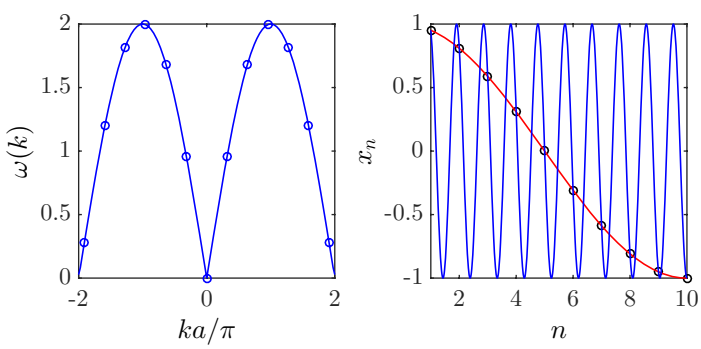

Figure 2.23: (code) Dispersion relation of a one-dimensional crystalline lattice consisting of 20 atoms.

The displacements of individual oscillators are now,

$$x_n(t) = na + Ae^{\imath nka - \imath\omega t} \ . \tag{2.161}$$

We need now to discuss, what are the possible values for k. First, since by adding 2π to the value ka we get the same result, we may concentrate on the region $-\pi < ka < \pi$, called the first *Brillouin zone*. And since the crystal is symmetric (we can reverse the

order of all oscillators), we can assume cyclic boundary conditions, $e^{\imath nka} = e^{\imath(N-n)ka}$, such that $(N-2n)ka/2\pi$ is an arbitrary integer number for any n, for example $n = 0$,

$$k = \frac{2\pi}{Na} \cdot \ell \ , \tag{2.162}$$

for $\ell \in \mathbb{N}$. To stay within the Brillouin zone, we set $\ell = -\frac{N}{2}, .., \frac{N}{2}$. That is, we have N possible values, which corresponds to just half the number of degrees of freedom.

Let us consider particular solutions. In the center of the Brillouin zone, $k = 0$ we have,

$$x_n(t) = na + Ae^{\imath\omega t} \ , \tag{2.163}$$

which corresponds to an in-phase oscillation of all oscillators. On the edge of the Brillouin zone, $k = \pm\pi/a$,

$$x_n(t) = na + A(-1)^n e^{\imath\omega t} \ , \tag{2.164}$$

which corresponds to a movement, where consecutive oscillators oscillate in anti-phase.

2.4.5.1 Waves in diatomic crystalline lattices

Many lattices are diatomic, that is, made of two species of atoms with different masses. For example, the NaCl salt crystal is a lattice alternating Na^+ and Cl^- ions. In analogy with the monoatomic lattice we establish the equations of motion,

$$\begin{aligned} \ddot{x}_n &= -\omega_x^2(x_n - y_{n-1}) - \omega_x^2(x_n - y_n) \\ \ddot{y}_n &= -\omega_y^2(y_n - x_{n+1}) - \omega_y^2(y_n - x_n) \ , \end{aligned} \tag{2.165}$$

with $\omega_{x,y} \equiv \sqrt{k/m_{x,y}}$. Inserting the ansätze $x_n = Ae^{\imath(nka-\omega t)}$ and $y_n = Be^{\imath(nka-\omega t)}$, we find the equations,

$$\begin{aligned} -\omega^2 A &= -\omega_x^2(2A - Be^{-\imath ka} - B) \\ -\omega^2 B &= -\omega_y^2(2B - Ae^{\imath ka} - A) \ , \end{aligned} \tag{2.166}$$

or,

$$\begin{pmatrix} 2\omega_x^2 - \omega^2 & -\omega_x^2(1 + e^{-ika}) \\ -\omega_y^2(1 + e^{\imath ka}) & 2\omega_y^2 - \omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \ . \tag{2.167}$$

The characteristic equation is,

$$0 = \det \hat{M} = (2\omega_x^2 - \omega^2)(2\omega_y^2 - \omega^2) - \omega_x^2(1 - e^{-\imath ka})\omega_y^2(1 - e^{\imath ka}) \ , \tag{2.168}$$

with the solution,

$$\omega^2 = \omega_x^2 + \omega_y^2 \pm \sqrt{\omega_x^4 + \omega_y^4 + 2\omega_x^2\omega_y^2 \cos ka} \ . \tag{2.169}$$

For $ka \ll 1$ we can approximate,

$$\begin{aligned} \omega^2 &\simeq \omega_x^2 + \omega_y^2 \pm \sqrt{(\omega_x^2 + \omega_y^2)^2 - \omega_x^2\omega_y^2 k^2 a^2} \\ &\simeq 2(\omega_x^2 + \omega_y^2) \ , \ \ \omega_x^2\omega_y^2 k^2 a^2 \ . \end{aligned} \tag{2.170}$$

The first eigenvalue is called the *optical branch* and the second the *acoustic branch*. The optical branch corresponds to an anti-phase motion of the atoms of the species x and y. This motion can be excited by light fields. The acoustic branch corresponds to an in-phase motion of the atoms.

In contrast, for $ka \simeq \pm\pi/a$ we obtain,

$$\omega^2 = \omega_x^2 \quad , \quad \omega_y^2 \ . \tag{2.171}$$

In these solutions either atom x oscillates while y stays at rest, or the opposite.

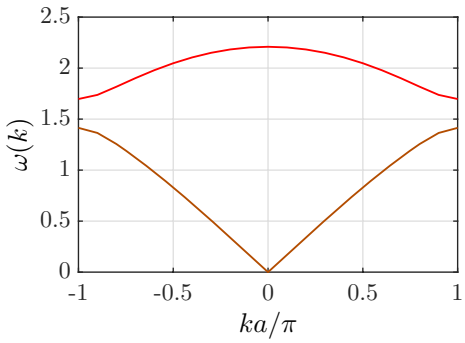

Figure 2.24: (code) Dispersion relation in a one-dimensional lattice showing in blue the optical branch and in green the acoustic branch.

2.4.6 Exercises

2.4.6.1 Ex: Fourier expansion

Expand the function $f(\xi) = \sin^3 \xi$ in a Fourier series.

2.4.6.2 Ex: Fourier expansion of sea waves

Surface waves on the sea are often better described by the function $f(x,t) = (kx - 2n\pi)^2$ inside the intervals $x \in [(2n-1)\pi/k, (2n+1)\pi/k]$ com $n \in \mathbb{N}$. Expands the wave in a spatial Fourier series. Use the formula $\int z^2 \cos(bz) dz = \frac{1}{b^3}[(b^2z^2 - 2)\sin bz + 2bz \cos bz]$.

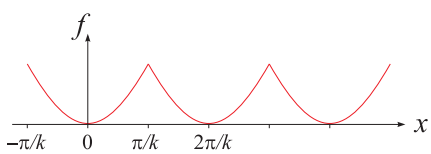

Figure 2.25: (code)

2.4.6.3 Ex: Fourier expansion of a rectified signal

An alternating electric current can be turned into a signal of half-cycles, $f(t) = |\cos\frac{\omega t}{2}|$, by a diode rectifier bridge. Expand this signal into a temporal Fourier series. Use the formula $\int \cos(az)\cos(bz)dz = \frac{\sin[(a-b)z]}{2(a-b)} + \frac{\sin[(a+b)z]}{2(a+b)}$.

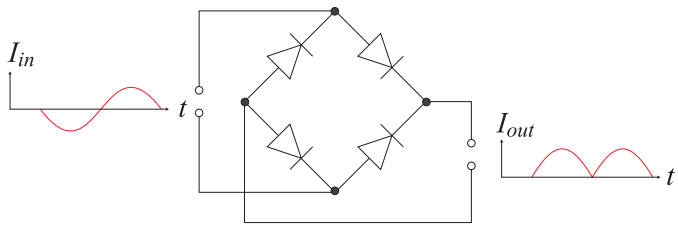

Figure 2.26: Fourier expansion of a rectified signal.

2.4.6.4 Ex: Action of a low pass filter on a spectrum

One method of creating a sinusoidal signal in electronics consist in first creating a rectangular signal via a switching circuit and then pass this signal through a low-pass filter by cutting off the harmonics. Simulate this procedure using the Fourier transform method starting from the rectangular signal $S(t) = \sin\omega t/|\sin\omega t|$ with $\omega/2\pi = 1\,\text{kHz}$ and using a low pass filter, such as $F(\omega) = 1/\left(1 + (\omega/\omega_g)^2\right)$, where the cut-off frequency is, $\omega_g/2\pi = 1\,\text{kHz}$. Evaluate the harmonic distortion of the rectangular signal and the filtered signal.

2.4.6.5 Ex: Tuning a violin

What would be the beat frequency between the pitches $3f_{c'}$ and $2f_{g'}$ if the strings were tuned logarithmically equidistant.

2.4.6.6 Ex: String instruments

Imagine a string instrument with 12 strings tuned in quints. How far would be the highest string from a harmonic of the lowest one.

2.4.6.7 Ex: String instruments

Prepare a list comparing the harmonics up to ninth order in the harmonic and in the tempered scale.

2.4.6.8 Ex: Frequency beating of sound waves

To tune a violin a musician first tunes the a-string ('la') at $f_a = 440\,\text{Hz}$ and then plays two neighboring strings, paying attention to the frequency beats. When playing the a- and the e-string ('mi'), the violinist hears a beat frequency of $3\,\text{Hz}$, and he notes that this frequency increases as the tension of the e-string increases. (The e-string is tuned to $f_e = 660\,\text{Hz}$.)

a. Why is there a beat when the two strings are played simultaneously?
b. What is the vibration frequency of the e-string when the beat frequency it generates together with the a-string is 3 Hz?
c. If the tension on the e-string is 80 N for a beat frequency of 3 Hz, what tension corresponds to a perfect tuning of the string?

2.4.6.9 Ex: Frequency beating of sound waves

A violinist tries to tune the strings of his instrument.
a. Comparing the a-string ('la') to a tuning fork ($\nu_{dia} = 440\,\text{Hz}$), he hears a beat with the frequency 1 Hz. By increasing the tension on the rope, the beat frequency increases. What was the frequency of the 'a'-string before the tension increased?
b. After having adjusted the a-string the violinist wants to tune the d-string ('re'). He realizes that the second harmonic $3\nu_d$ produces with the first harmonic of the a-string ($2\nu_a$) a beat of 1 Hz. Decreasing the tension of the d-string the beat disappears. What was the initial frequency of the d-string and by what percentage does the violinist need to decrease the tension of the string?

2.4.6.10 Ex: Normal modes on a string

A stretched wire of mass m, length L, and tension T is triggered by two sources, one at each end. Both sources have the same frequency ν and amplitude A, but are out of phase by exactly 180° with respect to each other. (At each end there is an anti-node.) What is the lowest possible value of ω consistent with the stationary vibrations of the wire?

2.4.6.11 Ex: Normal modes on a string

a. Find the total vibration energy of a wire of length L fixed at both ends and oscillating in its n-th characteristic mode with amplitude A. The tension on the wire is T, and its total mass is M. (**Suggestion:** Consider the integrated kinetic energy at the instant when the wire is straight.)
b. Calculate the total vibration energy of the same wire vibrating in the following superposition of normal modes:

$$Y(x,t) = A_1 \sin \tfrac{\pi x}{L} \cos \omega_1 t + A_3 \sin \tfrac{3\pi x}{L} \cos(\omega_3 t - \tfrac{\pi}{4}) \ .$$

You should be able to verify that it is the sum of the energies of the two modes taken separately.

2.4.6.12 Ex: Normal modes on a string

A wire of length L is attached at both ends under a tension T. The wire is pulled sideways by a distance h from its center, such that the rope adopts a triangular shape, and the it is released.
a. What is the energy of the subsequent oscillations. **Suggestion**: Consider the work that needs to be done against the tension to give the wire its initial deformation, and suppose that the tension remains unchanged upon a slight increase of its length caused by transverse the displacements.
b. How many times will the triangular shape reappear?

2.4.6.13 Ex: Waves on a rope

A string with linear mass density μ is attached at two points distant from each other by $L = 1\,\mathrm{m}$. A mass $m = 1\,\mathrm{kg}$ is now attached to one end of the rope that goes through a pulley, as shown in the figure. Excited by a vibrating pin with frequency $f = 1\,\mathrm{kHz}$ the string performs transverse vibrations with wavelength $\lambda = 2L$.
a. Calculate the propagation velocity of the wave.
b. At what frequency should the pin excite the rope to observe the third oscillation mode (three anti-nodes)?
c. Now the mass is doubled. Calculate the new speed of sound.
d. How should the mass be chosen to obtain a fundamental mode frequency equal to the frequency of the third mode calculated in (b)?

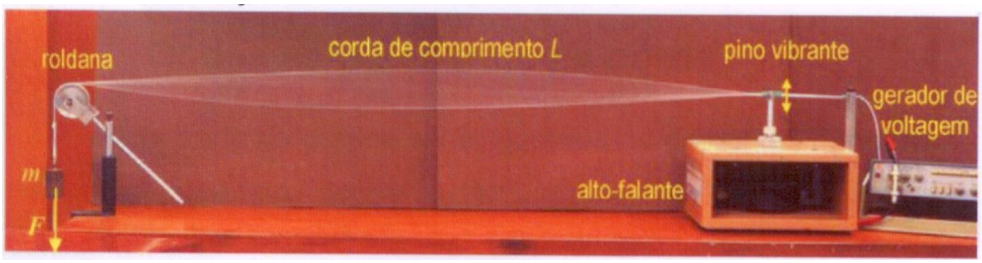

Figure 2.27: Waves on a rope.

2.5 Matter waves

Quantum mechanics tells us that light sometimes behaves like particles and matter like waves. Letting us guide by this analogy we will, in the following, guess the fundamental equations of motion for the propagation of matter waves from a comparison of the respective dispersion relations of light and massive particles.

2.5.1 Dispersion relation and Schrödinger's equation

On one hand, the propagation light is (in the vacuum) is described by the dispersion relation $\omega = ck$ or,

$$\omega^2 - c^2k^2 = 0 \ . \tag{2.172}$$

Since light is a wave it can, in the most general form, be described by a wavepacket, $A(\mathbf{r}, t) = \int e^{\imath(\mathbf{k}\cdot\mathbf{r}-\omega t)} a(\mathbf{k}) d^3k$. It is easy to verify that the *wave equation*,

$$\frac{\partial^2}{\partial t^2} A - c^2 \nabla^2 A = 0 \ , \tag{2.173}$$

reproduces the dispersion relation.

On the other hand, slow massive particles possess the kinetic energy,

$$E = \frac{p^2}{2m} \ . \tag{2.174}$$

With de Broglie's hypothesis that even a massive particle has wavelength, we can try an *ansatz* [10] for a wave equation satisfying the dispersion relation (2.174). From Planck's formula, $E = \hbar\omega$, and the formula of *Louis de Broglie*, $\mathbf{p} = \hbar\mathbf{k}$, describing the particle by a wavepacket not being subject to external forces $\psi(\mathbf{r}, t) = \int e^{\imath(\mathbf{k}\cdot\mathbf{r}-\omega t)}\phi(\mathbf{k})d^3k$, it is easy to verify that the differential equation,

$$\imath\hbar\frac{\partial}{\partial t}\psi = \left(-\frac{\hbar^2}{2m}\nabla^2\right)\psi \ , \tag{2.175}$$

reproduces the dispersion relation. If the particle is subject to a potential, its total energy is $E = \mathbf{p}^2/2m + V(\mathbf{r}, t)$. This dispersion relation corresponds to the famous *Schrödinger equation*,

$$\imath\hbar\frac{\partial}{\partial t}\psi = \left(-\frac{\hbar^2}{2m}\Delta + V(r, t)\right)\psi \ . \tag{2.176}$$

Since we accept that light particles and lenses behave like a wave, to calculate their trajectories, we must determine the potential landscape $V(\mathbf{r})$ in which this particle moves before solving the Schrödinger equation. This is the role of *wave mechanics*, which is one of the formulations of quantum mechanics.

2.5.1.1 Scalar waves and vectorial waves

The electromagnetic field is a *vector field*, since $\vec{\mathcal{E}}(\mathbf{r}, t)$ and $\vec{\mathcal{B}}(\mathbf{r}, t)$ are vectors. Therefore, it has a polarization. In contrast, the field of matter $\psi(\mathbf{r}, t)$ is a *scalar field* and therefore does not have the degree of freedom of polarization, in analogy with sound. This has important consequences, for example, the fact that two collinear light fields with orthogonal polarizations do not interfere has no analogue with matter wave fields.

2.5.2 Matter waves

Broglie's formula assigns a wave to each body, The wavelength decreases as the velocity of the particle grows. The necessity to describe a massive particle as a matter wave depends on the relationship between its Broglie wavelength and other characteristic quantities of the system under consideration. If the wavelength is large, we expect typical interference phenomena for waves; if the wavelength is small, the particle will behave like a mass, which is perfectly localized in space and incapable of interfering.

Characteristic features of the system may be, for example, the presence of a narrow slit diffracting the Broglie wave of an atom or an electron passing through it. Another characteristic feature is the average distance between several atoms. In fact, when an atomic gas is so cold, that is, composed of atoms so slow, that the Broglie wavelength of the atoms is longer than the average distance, then the atoms interfere with each other. In the case of bosonic atoms, the interference will be constructive, resulting in a matter wave of gigantic amplitude. This phenomenon is called Bose-Einstein condensation [11].

[10] Kick, work hypothesis, guess.
[11] See script on *Quantum mechanics* (2023).

Before calculating the temperature required for this phenomenon to happen, we need to inform the reader, that the interatomic distance can not be compressed arbitrarily, because below distances of typically $\bar{d} = 1\ \mu$m, the gas tends to form molecules. For the Broglie waves of different atoms to interfere, the wavelength must be longer. The average velocity of the atoms in a gas of temperature T is given by,

$$\frac{m}{2}\bar{v}^2 = \frac{k_B}{2}T\ .$$

Therefore, the temperature of the gas must be,

$$T = \frac{m\bar{v}^2}{k_B} = \frac{\bar{p}^2}{k_B m} = \frac{\hbar^2\bar{k}^2}{k_B m} = \frac{4\pi^2\hbar^2}{k_B m\lambda_{dB}^2} < \frac{h^2}{k_B m d^2}\ .$$

For rubidium atoms of mass $m = 87u$ we calculate $T < 200\,\text{nm}$.

The development of powerful experimental techniques allowed in 1995 the cooling of rubidium gases down to such low temperatures and the experimental realization of Bose-Einstein condensates, that is, matter waves made up of 10^6 atoms. See Exc. 2.5.3.1.

2.5.3 Exercises

2.5.3.1 Ex: Interference in Bose-Einstein condensates

Calculate the periodicity of the interference pattern of two Bose-Einstein condensates supposed to have intrinsic temperatures $T = 0$ interpenetrating at a relative velocity $v = 1\,\text{mm/s}$.

2.6 Further reading

H.M. Nussenzveig, Edgar Blucher (2014), *Curso de Física Básica: Fluidos, Vibrações e Ondas, Calor - vol 2* [ISBN]

Chapter 3

Gravitation

3.1 Planetary orbits

3.1.1 Kopernicus' laws

Nicolaus Copernicus published in 1543 his book *De revolutionibus orbium coelestium* in which he states:

1. The planetary orbit is a circle with epicycles.
2. The Sun is approximately at the center of the orbit.
3. The speed of the planet in the main orbit is constant.

Despite being correct in saying that the planets revolved around the Sun, Copernicus was incorrect in defining their orbits. It was Kepler who correctly defined the orbit of planets as follows:

1. The planetary orbit is not a circle with epicycles, but an ellipse.
2. The Sun is not at the center but at a focal point of the elliptical orbit.
3. Neither the linear speed nor the angular speed of the planet in the orbit is constant, but the area speed is constant.

3.1.2 Kepler's laws

Kepler's laws of planetary motion, published by *Johannes Kepler* between 1609 and 1619, describe the orbits of planets around the Sun:

1. The orbit of a planet is an ellipse with the Sun at one of the two foci.
2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
3. The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit.

The elliptical orbits of planets were indicated by calculations of the orbit of Mars. From this, Kepler inferred that other bodies in the Solar System, including those farther away from the Sun, also have elliptical orbits. The second law helps to establish

that when a planet is closer to the Sun, it travels faster. The third law expresses that the farther a planet is from the Sun, the slower its orbital speed, and vice versa. *Isaac Newton* showed in 1687 that relationships like Kepler's would apply in the Solar System as a consequence of his own laws of motion and law of universal gravitation. Do the Excs. 3.1.3.1, 3.1.3.2, and 3.1.3.3.

The eccentricity of the orbit of the Earth makes the time from the March equinox to the September equinox, around 186 days, unequal to the time from the September equinox to the March equinox, around 179 days. A diameter would cut the orbit into equal parts, but the plane through the Sun parallel to the equator of the Earth cuts the orbit into two parts with areas in a 186 to 179 ratio, so the eccentricity of the orbit of the Earth is approximately,

$$e \approx \frac{\pi}{4}\frac{186-179}{186+179} \approx 0.015 \ , \tag{3.1}$$

which is close to the correct value (0.016710218). The accuracy of this calculation requires that the two dates chosen be along the elliptical orbit's minor axis and that the midpoints of each half be along the major axis. As the two dates chosen here are equinoxes, this will be correct when perihelion, the date the Earth is closest to the Sun, falls on a solstice. The current perihelion, near January 4, is fairly close to the solstice of December 21 or 22.

3.1.3 Exercises

3.1.3.1 Ex: Kepler orbits

The moon moves in a good approximation on a circular path with radius $R = 384000\,\mathrm{km}$ around the Earth. Assume that the Earth's mass would suddenly decrease.
a. How much would the mass have to decrease so that the moon could escape the Earth?
b. How would the moon's orbit change if the mass decreased by a factor of 3, 2 or 1.5?

3.1.3.2 Ex: Kepler orbits of missiles

Consider an object of mass $m \ll M_{\oplus}$ which is launched at an initial velocity $\mathbf{v}_0$ (at an angle θ relative to the Earth's surface). We neglects any friction.
a. What possible trajectories can the object move on? How does the type of trajectory depend on the conservation parameters?
b. Calculate the maximum speed that the object may have to move on a closed trajectory. Does this speed depend on θ? Does the projectile always fall back to Earth when the path is closed?
c. Neglecting the Earth's rotation calculate the flight distance of the projectile above the Earth's surface for velocities below the above-mentioned limit velocity.
Help: Set the center of the Earth in the focal point of the Kepler orbit.

3.1.3.3 Ex: Halley's Comet

The comet Haley moves like a planet on an elliptical orbit around the sun. Its orbital period is 75 years and the closest distance to the sun is 0.5 AE. (One astronomical unit is the distance from the Earth to the sun, assuming that the orbit of the Earth around the sun is a circular orbit.)
a. Use this information to calculate the value for the major semi-axis a and the minor semi-axis b of the comet's orbit in astronomical units. Use this to determine the eccentricity ε of the orbit.
b. What is the maximum distance of the comet from the sun?
c. Calculate the minimum and maximum speed of the comet on its orbit.

3.2 Newton's law

Newton's law of *gravity* about the force between two massive bodies,

$$\mathbf{F} = -\nabla V(r) \ . \tag{3.2}$$

can be deduced from a conservative central potential,

$$\boxed{V(r) = \frac{\gamma_N M m}{r}} \ . \tag{3.3}$$

If $M = M_{♁}$ is the mass of the Earth, a test mass m close to the surface ($r_{♁} \approx 6378\,\mathrm{km}$) will be accelerated by,

$$g = \frac{F}{m} = -\frac{\partial}{\partial r}\frac{\gamma_N M_{♁}}{r}\bigg|_{r=R_{♁}} = \frac{\gamma_N M_{♁}}{R_{♁}^2} = 9.81\,\mathrm{m/s^2} \ . \tag{3.4}$$

with Newton's constant,

$$\boxed{\gamma_N = 6.67 \cdot 10^{-11}\,\mathrm{m^3/kg\ s^2}} \ . \tag{3.5}$$

3.2.1 Cosmic velocities

3.2.1.1 First cosmic velocity

The first *cosmic velocity* is defined as the velocity that a body must have in order to circle the center of the Earth on an orbit with the Earth's radius. We calculate this velocity from the condition that the centripetal force be equal to the centrifugal force,

$$\frac{m v_1^2}{r_{♁}} = \gamma_N \frac{m M_{♁}}{r_{♁}^2} \qquad \Rightarrow \qquad v_1 = \sqrt{\frac{\gamma_N M_{♁}}{r_{♁}}} \ , \tag{3.6}$$

yielding $v_1 \approx 7.91\,\mathrm{km/s} = 2.84 \cdot 10^4\,\mathrm{km/h}$. In Exc. 3.2.3.1 we estimate the mass of the milky way galaxy from the velocity of the sun and its distance from the galaxy's center. In Exc. 3.2.3.2 we compare the heights of stationary orbits around the Earth and the moon.

Example 17 (*Angular velocity of a satellite*): Here, we calculate the velocity of a satellite on a circular orbit at a height of 400 km above the Earth's surface,

$$v_1 = \sqrt{\frac{\gamma_N M_{\text{♁}}}{r_{\text{♁}} + h}} \ ,$$

yielding $v_1 \approx 7.66\,\text{km/s} = 2.76 \cdot 10^4\,\text{km/h}$.

3.2.1.2 Escape velocity

The *escape velocity* or second cosmic velocity is the velocity that a body must have to be able to leave the Earth's gravity field completely. We calculate the second cosmic speed for the Earth from,

$$E_{kin} = \frac{m}{2} v_2^2 = \text{final} - \text{initial energy in the limit final energy } \to 0 \ . \tag{3.7}$$

Hence,

$$E_{kin} = 0 - \left(-\gamma_N \frac{m M_{\text{♁}}}{r_{\text{♁}}}\right) \qquad \Rightarrow \qquad v_2 = \sqrt{\frac{2\gamma_N M_{\text{♁}}}{r_{\text{♁}}}} = v_1\sqrt{2} \ , \tag{3.8}$$

yielding $v_2 \approx 11.2\,\text{km/s} = 4.03 \cdot 10^4\,\text{km/h}$. Apparently, the cosmic velocities v_1 and v_2 are related. In Exc. 3.2.3.3 and 3.2.3.4 we calculate cosmic velocities for, respectively, Earth and the comet Tschurjumow-Gerasimenko.

Example 18 (*Escape velocity for a satellite*): The escape velocity for a satellite that is in a 400 km high orbit above the Earth's surface is $v_2 \approx 10.83\,\text{km/s} = 3.90 \cdot 10^4\,\text{km/h}$.

3.2.2 Deriving Kepler's laws from Newton's laws

3.2.2.1 Kepler's first law

The orbits are ellipses, with focal points F1 and F2 for the first planet and F1 and F3 for the second planet. The Sun is placed at focal point F1. The two shaded sectors A1 and A2 have the same surface area and the time for planet 1 to cover segment A1 is equal to the time to cover segment A2. The total orbit times for planet 1 and planet 2 have a ratio $(a_1/a_2)^{3/2}$.

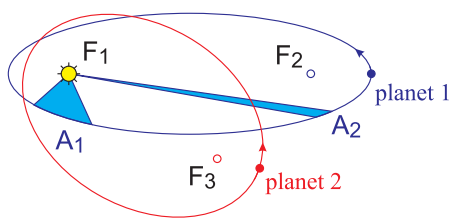

Figure 3.1: Illustration of Kepler's three laws with two planetary orbits.

3.2.2.2 Kepler's second law

The area swept by the planet's trajectory in infinitesimal time steps is,

$$A(t, t+dt) = \tfrac{1}{2}|\mathbf{r}(t) \times \dot{\mathbf{r}}(t)|dt = \tfrac{L}{2m}dt \ .$$

Since central potentials preserve angular momentum,

$$\dot{\mathbf{L}} = \frac{d}{dt} m\mathbf{r} \times \mathbf{p} = m(\dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}}) = m\mathbf{r} \times \ddot{\mathbf{r}} = -\mathbf{r} \times \nabla V(r) = -\mathbf{r} \times \frac{\partial V(r)}{\partial r}\hat{\mathbf{e}}_r = \mathbf{0} \ ,$$

for a given time difference $dt = t_1 - t_0$ the swept area is the same. Angular momentum is a constant of motion, $\dot{\mathbf{L}} = 0$, for central potentials.

3.2.2.3 Kepler's third law

3.2.3 Exercises

3.2.3.1 Ex: Mass of the Milky Way

Estimate the total mass of our galaxy (the milky way) using the parameters of the orbits of the sun (and the solar system) around the center of the galaxy. Assume that the major part of the mass of our galaxy is in the form of a uniform sphere (bulge). The speed of the sun on its way around the center of the galaxy is approximately $v = 250\,\mathrm{km/h}$, the distance of the sun from the center of the galaxy is approximately $r = 28000\,\mathrm{ly}$ (light years). To how many stars like our sun does this correspond to?

3.2.3.2 Ex: Gravitation on Earth and Moon

How high are the orbits of 'geo-stationary' and 'lunar-stationary' satellites?

3.2.3.3 Ex: Cosmic velocities

a. How long is the orbital period T of a 1 t satellite on a circular orbit at a height of 20 km around the Earth? How long is the orbital period T of the Earth around the sun (the mass of the sun is 3.334×10^5 times larger than that of the Earth)? At what distance from Earth is the orbit of a satellite geostationary?
b. Calculate the escape velocity from Earth (or cosmic speed) for a person weighing 75 kg.

3.2.3.4 Ex: Tschurjumow-Gerasimenko

The satellite Rosetta of the ESA ($m_{sat} = 3000\,\mathrm{kg}$) was placed on an orbit of the comet Tschurjumow-Gerasimenko (mass $m_{TG} = 3.14 \cdot 10^{12}\,\mathrm{kg}$, diameter $d_{TG} = 4\,\mathrm{km}$).
a. For the satellite to orbit the comet once a terrestrial day, what is the required height of the orbit?
b. What is the escape velocity from the comet's surface?

3.3 Gravitational potential

For an arbitrary mass distribution $\rho(\mathbf{r})$ the *gravitational potential* acting on a test mass m can be calculated from,

$$\boxed{V(\mathbf{r}) = -\gamma_N m \int_{\mathbb{R}^3} \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r'} \ . \tag{3.9}$$

For a point-mass with mass M located at the origin, $\mathbf{r}' = 0$, we parametrize $\rho(\mathbf{r}') = M\delta^3(\mathbf{r}')$, and recover Newton's law,

$$V(\mathbf{r}) = -\gamma_N \frac{Mm}{r} \ . \tag{3.10}$$

The gravitational potential being conservative, trajectories of test masses can simply be derived by solving the equation of motion,

$$\boxed{m\ddot{\mathbf{r}} = -\nabla V(\mathbf{r}) = \gamma_N m \int_{\mathbb{R}^3} \rho(\mathbf{r}') \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} d^3r'} \ . \tag{3.11}$$

If in practice analytic solution are beyond reach, numerical procedure are always possible.

Example 19 (*Gravitational potential in- and outside a homogeneous sphere*): In this example we will calculate the gravitational force that a particle of mass m is subjected to when placed inside a homogeneous sphere of radius R at a distance r from its center.
The potential exerted by a mass distribution with the density $\rho(\mathbf{r}')$ on a particle of mass m located at the position $\mathbf{r}$ is,

$$V(\mathbf{r}) = -\int \rho(\mathbf{r}') \frac{\gamma_N m}{|\mathbf{r}-\mathbf{r}'|} d^3r' = -\int_{sphere} \rho_0 \frac{\gamma_N m}{|\mathbf{r}-\mathbf{r}'|} r'^2 \sin\theta' dr' d\theta' d\phi' \ . \tag{3.12}$$

Substituting,

$$\xi \equiv |\mathbf{r}-\mathbf{r}'| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta'} \tag{3.13}$$
$$\frac{d\xi}{d\theta'} = \frac{rr'\sin\theta'}{\xi} \ ,$$

we obtain,

$$V(\mathbf{r}) = -\int_{sphere} \rho_0 \frac{\gamma_N m r'}{r} d\xi dr' d\phi' = \frac{2\pi\rho_0\gamma_N m}{r} \int_0^R \int_{\xi_{\min}}^{\xi_{\max}} r' d\xi dr' \ . \tag{3.14}$$

The integration limits follow from the values adopted by ξ for $\theta = 0$ resp. $\theta = \pi$. For $r \leq R$ we have that r' is always greater than r. Hence, $\xi = r' - r, .., r' + r$. For $R \leq r$ we have that r' is always smaller than r. Hence, $\xi = r - r', .., r' + r$.

$$V(\mathbf{r}) = -\frac{2\pi\rho_0\gamma_N m}{r} \begin{cases} \int_r^R 2rr'dr' + \int_0^r 2r'^2 dr' \\ \int_0^R 2r'^2 dr' \end{cases} \quad \text{for} \quad \begin{cases} r \leq R \\ R \leq r \end{cases} \ . \tag{3.15}$$

With the sphere's mass,

$$M = \frac{4\pi\rho_0 R^3}{3} \ , \tag{3.16}$$

the potential becomes,

$$V(\mathbf{r}) = -2\pi\rho_0\gamma_N m\left(R^2 - \frac{1}{3}r^2\right)\theta(R-r) - 2\pi\rho_0\gamma_N m\frac{2R^3}{3r}\theta(r-R) \tag{3.17}$$
$$= -\gamma_N Mm\left(\frac{3}{2R} - \frac{r^2}{2R^3}\right)\theta(R-r) - \gamma_N Mm\frac{1}{r}\theta(r-R)\ .$$

The force can be calculated using the gradient in spherical coordinates,

$$\mathbf{F} = -\nabla V(\mathbf{r}) = -\hat{\mathbf{e}}_r\frac{\partial}{\partial r}V(\mathbf{r}) \tag{3.18}$$
$$= -\hat{\mathbf{e}}_r\gamma_N Mm\frac{r}{R^3}\theta(R-r) - \hat{\mathbf{e}}_r\gamma_N Mm\frac{1}{r^2}\theta(r-R)\ .$$

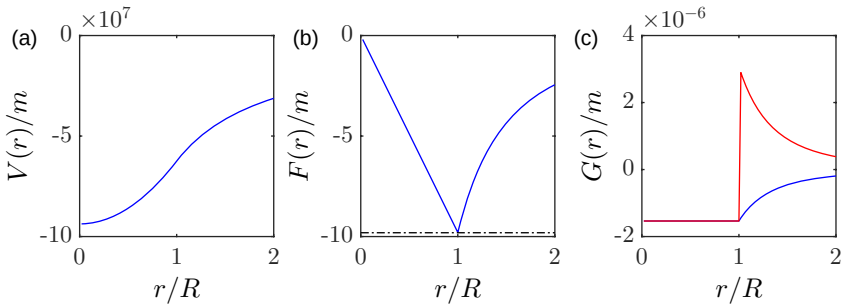

Figure 3.2: (code) Gravity in- and outside of Earth. (a) Gravitational potential, (b) gravitational force, and (c) radial (blue) and transverse (red) gravity gradient.

The above example shows that

1. outside a spherical mass distribution the gravitational potential can simply be replaced by that of a point mass sitting at the center of the mass distribution;

2. the superposition principle,

$$\boxed{V_{\rho_1+\rho_2}(\mathbf{r}) = V_{\rho_1}(\mathbf{r}) + V_{\rho_2}(\mathbf{r})}\ , \tag{3.19}$$

   allows us to described the impact of mass cavities via simple subtraction

In classical mechanics we often describe gravity as a homogenous force field, which can be derived from a potential scaling linearly with the height above normal ground,

$$V(h) = mgh\ . \tag{3.20}$$

Obviously, this is approximation obtained by linearizing the gravitational potential on the Earth's surface. From Newton's law,

$$V(\mathbf{r}) = -\frac{\gamma_N Mm}{r}\ , \tag{3.21}$$

using the Taylor expansion:

$$V(\mathbf{r}+\mathbf{h}) = e^{\mathbf{h}\cdot\nabla_\mathbf{r}} V(\mathbf{r}) = \sum_{\nu=0}^{\infty} \frac{(\mathbf{h}\cdot\nabla_\mathbf{r})^\nu}{\nu!} V(\mathbf{r}) = V(\mathbf{r}) + (\mathbf{h}\cdot\nabla_\mathbf{r})V(\mathbf{r}) + \frac{1}{2}(\mathbf{h}\cdot\nabla_\mathbf{r})(\mathbf{h}\cdot\nabla_\mathbf{r})V(\mathbf{r}) \ , \tag{3.22}$$

we get,

$$V(\mathbf{r}+\mathbf{h}) \simeq V(\mathbf{r}) + h\frac{\gamma_N M m}{r^2} = V(\mathbf{r}) + hgm \ . \tag{3.23}$$

In Exc. 3.3.5.1 we derive an expression generalizing Eq. (3.17) to arbitrary isotropic gravitational potentials. In Excs. 3.3.5.2, 3.3.5.3, 3.3.5.4, and 3.3.5.5 we calculate the potentials for other isotropic mass distributions. In Exc. 3.3.5.6 we use the superposition principle to calculate the potential generated by a spherical cavity inside a homogeneous sphere. In Excs. 3.3.5.7, 3.3.5.8, and 3.3.5.9 we calculate potentials generated by non-spherical density distributions. In Excs. 3.3.5.10, 3.3.5.11, and 3.3.5.12 we apply the results derived for the Earth's inner gravitational potential to derive possible trajectories through boreholes traversing the Earth.

3.3.1 Rotation and divergence of gravitational force fields

The rotation and divergence of gravitational force fields are,

$$\begin{aligned} \nabla^2 V(\mathbf{r}) &= \nabla\cdot\mathbf{F}(\mathbf{r}) = -4\pi\gamma_N m\rho(\mathbf{r}) \\ \nabla\times\mathbf{F}(\mathbf{r}) &= 0 \ . \end{aligned} \tag{3.24}$$

The integral formulation of Eq. (3.24) reads,

$$\oint_{\partial V} \mathbf{F}\cdot d\mathbf{S} = -4\pi\gamma_N M m \ , \tag{3.25}$$

with $M = \oint_{\partial V} \rho(\mathbf{r}) d^3r$.

The interpretation of these expressions are:

- The Poisson equation relates the divergence of the force field directly to the density distribution.
- The divergence is nothing else than the diagonal of gravity gradient defined in Sec. 3.3.2.
- For being conservative, gravitational potentials are rotation-free.
- The integral over a closed surface is proportional to the enclosed mass.

The Lagrangian density for Newtonian gravity is,

$$\mathcal{L}(\mathbf{r},t) = -\rho(\mathbf{r},t) - \tfrac{1}{8\pi\gamma_N}[\nabla V(\mathbf{r},t)]^2 \ . \tag{3.26}$$

Applying the Hamitonian principle to this Lagrangian one recovers the Poisson equation for gravity.

3.3.2 Gravity gradients

The *gravity gradient* is a tensor defined as the second derivative of the potential,

$$\boxed{G_{kl}(\mathbf{r}) = G_{lk}(\mathbf{r}) = \frac{\partial g_l(\mathbf{r})}{\partial x_k} = \frac{1}{m}\frac{\partial F_l(\mathbf{r})}{\partial x_k} = -\frac{1}{m}\frac{\partial}{\partial x_k}\frac{\partial V(\mathbf{r})}{\partial x_l}} \ . \tag{3.27}$$

Inserting the potential (3.9) we obtain,

$$\begin{aligned} G_{kl}(\mathbf{r}) &= -\frac{1}{m}\frac{\partial}{\partial x_k}\frac{\partial V(\mathbf{r})}{\partial x_l} = \gamma_N \int_{R^3} \rho(\mathbf{r}')\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l}\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3r' \qquad (3.28)\\ &= \gamma_N \int_{R^3} \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^5}\begin{pmatrix} 3(x-x')^2-(\mathbf{r}-\mathbf{r}')^2 & 3(x-x')(y-y') & 3(x-x')(z-z') \\ 3(x-x')(y-y') & 3(y-y')^2-(\mathbf{r}-\mathbf{r}')^2 & 3(y-y')(z-z') \\ 3(x-x')(z-z') & 3(y-y')(z-z') & 3(z-z')^2-(\mathbf{r}-\mathbf{r}')^2 \end{pmatrix} d^3r' \\ &= \gamma_N \int_{R^3} \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} K_{kl}(\mathbf{r}-\mathbf{r}')d^3r' \ , \end{aligned}$$

defining the kernel,

$$K_{kl}(\mathbf{r}-\mathbf{r}') \equiv \frac{3(x_k-x_k')(x_l-x_l')-\delta_{kl}(\mathbf{r}-\mathbf{r}')^2}{|\mathbf{r}-\mathbf{r}'|^2} \ . \tag{3.29}$$

For example, for the gravitational potential generated by a point mass,

$$V(r) = \gamma_N \frac{Mm}{r} = \gamma_N \frac{Mm}{\sqrt{x^2+y^2+z^2}} \ , \tag{3.30}$$

we find,

$$\begin{aligned} G_{kl}(\mathbf{r}) = -\frac{1}{m}\frac{\partial}{\partial x_k}\frac{\partial V(\mathbf{r})}{\partial x_l} &= \frac{\gamma_N M}{r^5}\begin{pmatrix} 3x^2-r^2 & 3xy & 3xz \\ 3xy & 3y^2-r^2 & 3yz \\ 3xz & 3yz & 3z^2-r^2 \end{pmatrix} \qquad (3.31)\\ &= \frac{\gamma_N M}{r^5}(3x_kx_l - r^2\delta_{kl}) \ . \end{aligned}$$

Gravity gradients are often given in units of $1\,\text{Eotvos} = 10^{-9}\,\text{s}^{-2}$. In Exc. 3.3.5.13 we calculate the gravity gradient tensor of Earth (modeled as an idealized sphere) at the north-pole.

Example 20 (*Gravitational curvature in- and outside a homogeneous sphere*): The gravitational potential an force in- and outside a homogeneous sphere have been calculated in the example 19. Using the result we derive the gravity gradient,

$$\begin{aligned} G_{kl}(\mathbf{r}) &= -\frac{1}{m}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l}V(\mathbf{r}) \qquad (3.32)\\ &= -\frac{\gamma_N M}{R^3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\theta(R-r) + \frac{\gamma_N M}{r^5}\begin{pmatrix} 3x^2-r^2 & 3xy & 3xz \\ 3xy & 3y^2-r^2 & 3yz \\ 3xz & 3yz & 3z^2-r^2 \end{pmatrix}\theta(r-R) \\ &= -\frac{\gamma_N M}{R^3}\delta_{kl}\theta(R-r) - \frac{\gamma_N M}{r^3}\left(\delta_{kl} - \frac{3x_kx_l}{r^2}\right)\theta(r-R) \ . \end{aligned}$$

The example 19 revealed that neither the potential nor the force are discontinuous at the sphere's surface. In contrast, the radial component of the curvature $G_{k=l}(r=R)$ is discontinuous at the north pole, while the transverse components $G_{k\neq l}(r=R)$ stay continuous, which is obviously due to the isotropic symmetry of the potential. To see this better, let us move along the symmetry axis setting $\mathbf{r} = r\hat{\mathbf{e}}_z$,

$$G_{kl}(r\hat{\mathbf{e}}_z) = -\frac{\gamma_N M}{R^3}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\theta(R-r) - \frac{\gamma_N M}{r^3}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix}\theta(r-R)\ . \tag{3.33}$$

Applying this results to Earth, be find inside Earth a constant gravity gradient of $-\gamma_N M_{\text{♁}}/R_{\text{♁}}^3 = 1.54\cdot 10^{-6}\,\mathrm{s}^{-2}$.

3.3.2.1 Gravimetry and gravity gradiometry

Gravity-gradiometers measure spatial variations of the gravitational acceleration. Being obtained as second derivatives of the gravitational potential, they are more sensitive to local mass variations, as nearly homogeneous large scale contributions to the acceleration are removed. For this reason, gravity-gradiometers need to be less accurate, provided they are sensitive enough. In Exc. 3.3.5.14 we estimate the sensitivity of modern gravimeters.

Example 21 (*Gravitation in- and outside a massive shell*): The calculations of examples 19 and 20 can be generalized for a homogenous massive shell with density ρ_1, inner radius R_i, and outer radius R_0. In Exc. 3.3.5.2 we show that the gravitational potential is,

$$\begin{aligned} V(\mathbf{r}) = -2\pi\rho_1\gamma_N m\Big[&(R_o^2 - R_i^2)\theta(R_i - r) \\ &+\left(R_o^2 - \frac{r^2}{3} - \frac{2R_o^3}{3r}\right)\theta(r-R_i)\theta(R_o-r) + \frac{2(R_o^3-R_i^3)}{3r}\theta(r-R_o)\Big]\ , \end{aligned} \tag{3.34}$$

the gravitational force,

$$\begin{aligned} \mathbf{F}(\mathbf{r} = -\nabla V(\mathbf{r}) &= -\hat{\mathbf{e}}_r\frac{\partial}{\partial r}V(\mathbf{r}) \\ &= \hat{\mathbf{e}}_r\frac{4\pi\rho_1\gamma_N m}{3}\left[\left(-r + \frac{R_o^3}{r^2}\right)\theta(r-R_i)\theta(R_o-r) - \frac{R_o^3-R_i^3}{r^2}\theta(r-R_o)\right]\ , \end{aligned} \tag{3.35}$$

and the gravity gradient,

$$\begin{aligned} G_{kl}(\mathbf{r}) &= -\frac{1}{m}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l}V(\mathbf{r}) \\ &= -\frac{4\pi\rho_1\gamma_N}{3}\left[\left(\delta_{kl} - \frac{R_i^3}{r^3}\left(\delta_{kl} - \frac{3x_k x_l}{r^2}\right)\right)\theta(r-R_i)\theta(R_o-r)\right. \\ &\quad\left. + \frac{R_o^3-R_i^3}{r^3}\left(\delta_{kl} - \frac{3x_k x_l}{r^2}\right)\theta(r-R_o)\right]\ . \end{aligned} \tag{3.36}$$

For $R_i \to 0$ we recover the results of example 20. Particularly along the symmetry axis,

$$G_{zz}(\mathbf{r}) = -\frac{4\pi\rho_1\gamma_N}{3}\left[\left(1 + \frac{2R_i^3}{r^3}\right)\theta(r-R_i)\theta(R_o-r) - 2\frac{R_o^3-R_i^3}{r^3}\theta(r-R_o)\right]\ . \tag{3.37}$$

3.3.3 Constants of motion

Trajectories can also be derive exploiting constants of motion. In Excs. 3.3.5.17 to 3.3.5.22 we calculate trajectories of bodies under the influence of gravity.

3.3.4 The virial law

The virial law states,

$$\boxed{\overline{T} = -\overline{\frac{1}{2}\sum_i \mathbf{F}_i \mathbf{r}_i}} \ . \tag{3.38}$$

For potentials of the form $V(r) = \alpha r^k$ we have,

$$\mathbf{F} = -\nabla V = -k\alpha r^{k-1}\hat{\mathbf{e}}_r \ . \tag{3.39}$$

Thus $\overline{T}$ and $\overline{V}$ related via,

$$\overline{T} = -\overline{\frac{1}{2}\sum_i \mathbf{F}_i \mathbf{r}_i} = \frac{k}{2}\overline{\sum_i \alpha r_i^{k-1}\hat{\mathbf{e}}_{r_i}\cdot\mathbf{r}_i} = \frac{k}{2}\overline{\sum_i \alpha r_i^k} = \frac{k}{2}\overline{V} \ . \tag{3.40}$$

In Exc. 3.3.5.23 we apply the virial law to a spring pendulum.

Example 22 (*The virial law for the harmonic potential and for* $1/r$-*potentials*): The special case $k = 2$ yields,

$$V(r) = \alpha r^2 \Rightarrow \overline{T} = \overline{V} \ , \tag{3.41}$$

and corresponds to a harmonic oscillator with $\alpha = \frac{1}{2}m\omega_0^2$.
The special case $k = -1$ yields,

$$V(r) = \frac{\alpha}{r} \Rightarrow \overline{T} = -\frac{1}{2}\overline{V} \ , \tag{3.42}$$

and corresponds to a Coulomb potential with $\alpha = \frac{q_1 q_2}{4\pi\epsilon_0}$, respectively a gravitational potential with $\alpha = -\gamma_N Mm$.
In the case of the gravitational potential, for positive total energy, we get,

$$E = \overline{T} + \overline{V} > 0 \qquad \overline{T} = -\frac{1}{2}\overline{V} \Rightarrow E = \frac{1}{2}\overline{V} > 0 \Rightarrow Mm < 0 \ . \tag{3.43}$$

Thus,

$$\overline{T} = \frac{1}{2}m\overline{v^2} = -\frac{1}{2}\overline{V} < 0 \Rightarrow m < 0 \tag{3.44}$$

This leads to the demand for negative masses, which is not sensible. The virial theorem can only apply to bound systems with $E < 0$.

3.3.5 Exercises

3.3.5.1 Ex: Arbitrary isotropic mass density distributions

a. Generalize the calculation of gravitational potentials and forces exhibited in example 18 to arbitrary, but isotropic mass density distributions $\rho(\mathbf{r}') = \rho(r')$.
b. Study the case of a sharp edge, $\rho(r') \equiv \rho(r')\theta(R - r')$.
c. Study the case of a homogeneous distribution, $\rho(r') \equiv \rho_0\theta(R - r')$, for a sphere with total mass M.
d. Study the case of a parabolic distribution, $\rho(r') \equiv \rho_0\left(1 - \frac{r'^2}{R^2}\right)\theta(R - r')$, for a sphere with total mass M.

3.3.5.2 Ex: Gravitational potential of a spherical shell

Consider a spherical shell with an inner radius a and an outer radius b.
a. Calculate the gravitational potential inside the sphere, inside the shell material and outside the sphere. (**Help:** Substitute the distance between the test particle m and a point of the mass distribution and make a case distinction for the integration limits for this distance variable.)
b. Calculate the force on a test particle.
c. Specify now for a massive sphere.
d. Specify for a very thin spherical shell.

3.3.5.3 Ex: Two concentric shells

Let us consider two concentric spherical shells of uniform density with masses M_1 and M_2. Calculate the force on a particle of mass m placed (a) inside the inner shell, (b) outside the inner but inside the outer shell, and (c) outside the outer sphere.

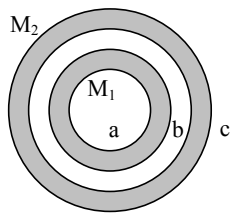

Figure 3.3:

3.3.5.4 Ex: Gravity influenced by a thin surface layer

Model the Earth as a homogeneous sphere of mass density ρ_0 isotropically covered by a $\Delta R = 1$m thick homogeneous layer with different density ρ_1. How does the gravitational potential depend on the ratio ρ_1/ρ_0?

3.3.5.5 Ex: Gravitational force inside a shell

Show through geometric arguments that a particle of mass m placed inside a spherical shell of uniform mass density is subject to zero force, regardless of the position of the particle. What would happen if the surface mass density was not constant?

3.3.5.6 Ex: Gravitational potential of a massive sphere with spherical cavity

A spherical cavity is machined into in a lead sphere of radius R such that its surface touches the outer surface of the massive sphere and passes through the its center. The primitive mass of the lead sphere is M. What will be the force that the sphere with the cavity will exert on a mass m at a distance z from the center of the outer sphere, when the mass and the centers of the sphere and the cavity are aligned?

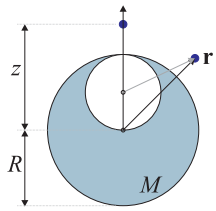

Figure 3.4: Scheme of the problem.

3.3.5.7 Ex: Gravitational potential of a disk

Calculate the potential of a homogeneous thin disc with the surface density $\sigma = M/\pi R^2 = \rho dz$ along the axis of symmetry and the gravitational force it exerts on a mass m.
Help: When integrating over the thickness a of the disk, use the relation: $\int_0^d zf(z')dz' = f(0)dz$.

3.3.5.8 Ex: Gravitational force of a ring

Calculate the gravitational force of a ring of linear mass density $\lambda = M/2\pi R = \rho dRdz$ on the symmetry axis.
Help: When integrating on the thickness of the ring, use the relations: $\int_0^{dz} f(z')dz' = f(0)dz$ and $\int_R^{R+dR} f(r')dr' = f(R)dR$.

3.3.5.9 Ex: Gravitational oscillation through a ring

Consider a heavy ring of mass M and radius R and a particle of mass m placed in its center. What is the frequency for small amplitude oscillations in the direction perpendicular to the plane of the ring?

3.3.5.10 Ex: Intraplanetary oscillation

A body of mass m is placed at a distance r_0 from the center of a planet of mass M and radius R.
a. Calculate the potential energy for $0 \leq r \leq \infty$. Suppose that the mass density of the planet is uniform and that the mass m can move within it through a tunnel. Consider $V(\infty) = 0$. Calculate the velocity as a function of r for $r < R$ knowing that $V(r_0) = 0$.

3.3.5.11 Ex: Shortcut avoiding the Earth's center

Show that in a tunnel dug through the Earth (not necessarily along a diameter) the movement of an object will be harmonic.

3.3.5.12 Ex: Shortcut through the Earth

a. Two innovative companies make suggestions on how to get mail to New Zealand as quickly as possible. One company suggests drilling a hole through the Earth, placing

the mail in a fireproof box and allowing it to swing through the hole (smoothly) through the center of the Earth so that it can be easily received by the recipient in New Zealand. The other company wants to shoot the mail in a very low orbit of only 1 m above the surface of the Earth at the first cosmic speed (smoothly) to New Zealand, where it should then be caught by a correspondingly soft pillow. Which of these two suggestions (if they were feasible) would get the mail faster to destination?
b. Assume that the well was planned incorrectly and that the hole missed the center of the earth by 100 km. What does the equation of motion look like?
Help: The mass distribution of the earth can be assumed to be homogeneous. Earth rotation and friction effects are neglected.

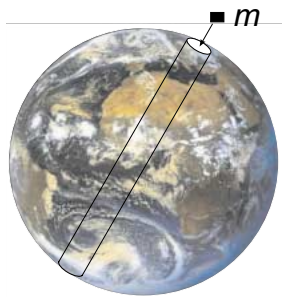

Figure 3.5:

3.3.5.13 Ex: Gravity gradient caused by underground cavities

In this exercise we discuss whether gravity gradiometry can identify the presence of underground cavities. We proceed in steps:
a. Assuming a homogeneous density distribution for Earth, calculate the gravity gradient tensor at the north-pole.
b. How does the tensor change in the presence of a point-like mass $M_1 = 10\,\text{tons}$ located at a distance $d = 1\,\text{m}$ in southern direction.
c. Describe the underground cavity by a spherical void centered at 1 m below the north-pole's surface at having a radius such that the missing mass corresponds to 10 tons of Earth material.

3.3.5.14 Ex: Gravity gradients

a. Modern commercial gravity gradiometers can measure acceleration gradients on the order of $|\nabla a| \approx 10^{-5}\,\text{s}^{-2}$. Compare with the gravity gradient on the Earth's surface. What is the smallest height difference detectable by a state of the art gradiometer?
b. Calculate the gravity gradient caused by a massive sphere of mass $m_{sphere} = 10\,\text{t}$ at $d = 1\,\text{m}$ distance?
c. The French company μQuans offers atomic quantum gravimeters with guaranteed sensitivities of $50\,\mu\text{Gal}/\sqrt{\text{Hz}}$ at a cycling frequency of 2 Hz. Assuming the Earth as a homogeneous sphere. For how long must the signal be integrated to be able to measure a 1 cm height variation over the Earth's surface.

d. For how long must the signal be integrated to be able to measure a gravity variation caused by a 10 t mass at 1 m distance.

3.3.5.15 Ex: Acceleration of a mass subject to a circular motion in an inhomogeneous force field

Commercial *Gravity Gradient Instruments* (GGI) are based on accelerometers mounted on the border of a disk of radius R rotating at a frequency ω. Let us suppose that the disk's rotation axis is the z-axis and that is located inside an inhomogeneous force field (e.g. gravity) characterized by its gradient tensor (assumed to be constant over time and over the length scale of R).
a. Calculate the time-dependent acceleration recorded by the accelerometer in radial direction.
b. The voltage signal delivered by the accelerometer is now added to one delivered by a second accelerometer sitting on the opposite side of the disk.
c. Finally, the signals are demodulated at 2ω and time-averaged over a period $2\pi/\omega$.

3.3.5.16 Ex: Angular momentum in spherical coordinates

a. Calculate the acceleration, the angular momentum, and its derivative in spherical coordinates using the result of Exc. ??.
b. Set $\theta = \frac{\pi}{2}$ in all expressions.
c. Derive the equation of motion for a central potential.

3.3.5.17 Ex: Scattering at a central force, angular momentum

Consider the scattering of a particle of mass M at an attractive central force field $\mathbf{F}(\mathbf{r}) = -\frac{\alpha}{r^2}\hat{\mathbf{e}}_r$ with $\alpha > 0$. Far from the force center the velocity of the particle is given by v_∞. The asymptotic distance perpendicular to the velocity for very large distances from the force center is called the impact parameter b.
a. Determine the relationship between the impact parameter b and the angular momentum L of the particle.
b. The path of the particle has the shape of a conic section, which in plane polar coordinates can be parametrized by $r = P/(1 - \epsilon\cos\phi)$. Find ϵ and P as a function of b, v_∞, M and α. c. Find an expression for $\sin(\theta/2)$. Here, θ is the scattering angle between the asymptotic orbits of the particle, i.e. the paths of the incoming and outgoing particles for large distances from the force center.
d. How does θ for constant v_∞ depend on the impact parameter b? Discuss the special cases $b = 0$ and $b \to \infty$.

3.3.5.18 Ex: Gravitational force, trajectory

The trajectory of the Kepler problem can be derived from the integral expression:

$$\varphi(r) = \varphi_0 + \int_{r_0}^{r} \frac{l\,dr}{r^2\sqrt{2m(E + \frac{\alpha}{r}) - \frac{l^2}{r^2}}} \ .$$

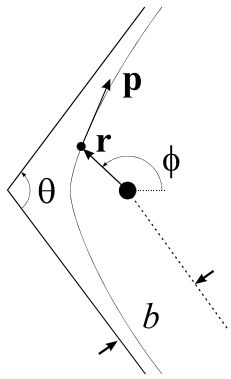

Figure 3.6:

Here E is the total energy, $\alpha = \gamma_N mM$ and $l = mr^2\dot{\varphi}$. We also introduce the quantities:

$$p = \frac{l^2}{m\alpha} \qquad \text{and} \qquad \varepsilon = \sqrt{1 + \frac{2El^2}{m\alpha^2}} \ .$$

a. Convince yourself that, with the substitution $\xi = (p/r - 1)/\varepsilon$, the integral expression can be written in the form:

$$\varphi(r) = \varphi_0 - \int_{\frac{1}{\varepsilon}(\frac{p}{r_0}-1)}^{\frac{1}{\varepsilon}(\frac{p}{r}-1)} \frac{d\xi}{\sqrt{1-\xi^2}} \ .$$

b. Show with the help of energy conservation that the minimum distance from the force center is determined by $r_{\min} = \frac{p}{1+\varepsilon}$ for all values of E.
c. Show that the trajectories for $\varphi_0 = \pi$ and $r = r_{\min}$ are $r(\varphi) = \frac{p}{1-\varepsilon\,\cos\varphi}$, where $\int dx/\sqrt{1-x^2} = \arcsin x$.
d. Confirm that for elliptical trajectories ($0 \leq \varepsilon < 1$ and $p = b^2/a^2$ where $a(b)$, the major semi-axis follows Kepler's 3rd law. Use the area theorem.

3.3.5.19 Ex: Central force, trajectory

Consider two masses m_1 and m_2 located at $\mathbf{r}_1$ and $\mathbf{r}_2$. There is an attractive force between them of the amount $F(\mathbf{r}_1, \mathbf{r}_2) = 2\lambda/|\mathbf{r}_1 - \mathbf{r}_2|^3$ ($\lambda > 0$).
a. Specify the angular momentum of the relative motion $\mathbf{l}$ and the energy conservation as a function of $\mathbf{r}$, $\mathbf{p}$ and the reduced mass μ, whereby we may designate by $E > 0$ the total energy of the system.
b. At the time $t = 0$ we let the relative distance of both particles be $r_{\min}$, the relative velocity in the direction of r be zero and $\varphi(r_{\min}) = 0$. Determine the relationship between $r_{\min}$, E, l, λ and μ. Is it possible to eliminate l and λ from the energy conservation law? Calculate the function $r(t)$.
c. Express $\frac{d}{d\varphi}r(\varphi) = \dot{r}/\dot{\varphi}$ as a function of E, l, r and $r_{\min}$ and calculate the trajectory $r(\varphi)$.

3.3.5.20 Ex: Ballistic movement

Consider the movement of an intercontinental missile launched at an inclination of θ_0, as shown in the figure, with speed v_0, in the indicated position. Calculate the body's trajectory.

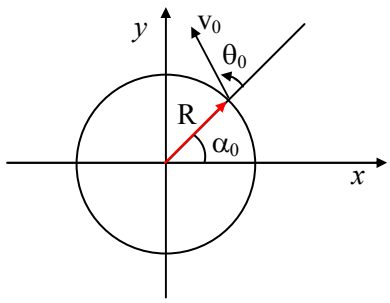

Figure 3.7:

3.3.5.21 Ex: Rotation of three bodies

Three identical bodies of mass M are located at the vertices of an equilateral triangle with border length L. How fast should they move, if they all rotate under the influence of mutual gravity, on a circular orbit that circumscribes the triangle always kept equilateral?

3.3.5.22 Ex: Rotation of two bodies

Consider two masses m and $2m$ with gravitational attraction. At what angular velocity should they rotate so that the distance d between them is constant?

3.3.5.23 Ex: The virial law

Consider a mathematical spring pendulum with $D = 100\,\mathrm{N/m}$ and an attached mass of $m = 100\,\mathrm{g}$. The average kinetic energy of the pendulum be $\overline{T} = 0.5\,\mathrm{J}$. What is the mean deflection $\overline{x}$ and the mean quadratic deflection $\overline{x^2}$?

3.4 Outlook on general relativity

The fundamental idea of *general relativity* is the *equivalence* of inert and heavy mass. While special relativity follows from Lorentz invariance, general relativity follows from Lorentz boost invariance, see also Secs. ?? and ??.

Example 23 (*Relativistic correction to Newton's law*): .

3.4.1 Gravitational red-shift

The *gravitational red-shift* $\Delta\omega$ suffered by a clock of mass m can be estimated from (see Sec.??),

$$\boxed{\hbar\Delta\omega = m\Delta\frac{V(\mathbf{r})}{m}} \ , \tag{3.45}$$

where $\Delta V(\mathbf{r})$ is the gravitational potential difference with and without a nearby heavy mass. The mass of the clock is a measure of its pace: $m = E/c^2 = \hbar\omega/c^2$. For instance, on the surface of Earth we get,

$$\hbar\Delta\omega = mg\Delta z = \frac{E}{c^2}g\Delta z = \frac{\hbar\omega}{c^2}g\Delta z \ . \tag{3.46}$$

Hence,

$$\frac{\Delta\omega}{\omega} = \frac{g}{c^2}\Delta z \simeq \Delta z \cdot 10^{-16}\,\mathrm{m}^{-1} \ . \tag{3.47}$$

3.4.2 Exercises

3.5 Further reading

H.M. Nussenzveig, Edgar Blucher (2013), *Curso de Física Básica: Mecânica - vol 1* [ISBN]

Chapter 4

Appendices to 'Classical Mechanics'

4.1 Constants and units in classical physics

4.1.1 Constants

4.1.1.1 Mathematical constants

π constant	$\pi = 3.1415..$
Euler constant	$e = 2.71828..$

4.1.1.2 Constants of the SI unit system

These numbers of the special adjustment CODATA 2019 were proposed as *exact* values.

frequency of the hyperfine transition of Cs	$\nu = 9\ 192\ 631\ 770\,\mathrm{Hz}$
velocity of light	$c = 299\ 792\ 458\,\mathrm{m/s}$
Planck's constant	$h = 6.626\ 070\ 15 \cdot 10^{-34}\,\mathrm{Js}$
electronic charge	$e = 1.602\ 176\ 634 \times 10^{-19}\,\mathrm{C}$
Boltzmann's constant	$k_B = 1.380\ 649 \times 10^{-23}\,\mathrm{J/K}$
Avogadro's constant	$N_A = 6.022\ 14076 \times 10^{23}\,\mathrm{mol}^{-1}$
Luminous efficiency	$K_{cd} = 683\,\mathrm{lm}$

4.1.1.3 Derived constants

fine-structure constant	$\alpha = e^2/4\pi\varepsilon_0\hbar c \approx 1/137$
vacuum permittivity	$\varepsilon_0 = 1/\mu_0 c^2 = 8.8542 \times 10^{-12}\,\mathrm{As/Vm}$
vacuum permeability	$\mu_0 = 10^{-7}\,\mathrm{Vs/Am}$
Faraday's constant	$F = 96485.309\,\mathrm{C/mol}$
atomic mass unit	$u_A = 1/N_A \times 1\mathrm{g/mol} = 1.6605402 \times 10^{-27}\,\mathrm{kg}$
gas constant	$R = N_A k_B = 8.314510\,\mathrm{L/mol\ K}$
Bohr radius	$a_B = \alpha/4\pi R_\infty = 0.529 \times 10^{-10}\,\mathrm{m}$
Bohr magneton	$\mu_B = e\hbar/2m_e = 9.27 \times 10^{-24}\,\mathrm{J/T}$
classical electron radius	$r_e = \alpha^2 a_B$
Rydberg constant	$R_\infty = m_e c\alpha^2/2h = 13.7\,\mathrm{eV}$
Compton wavelength	$\lambda_C = h/m_e c$
Thomson cross section	$\sigma_e = (8\pi/3)r_e^2$
gravitational constant	$\gamma = 6.67259 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$

4.1.1.4 Particle constants

electron mass	$m_e = 9.1096 \times 10^{-12}\,\mathrm{kg}$
g-factor of the electron	$g = 2.002\ 319\ 304\ 386$
muon mass	$m_\mu = 105.658389\,\mathrm{MeV}$
proton mass	$m_p = 938.27231\,\mathrm{MeV}$
g-factor of the proton	$g = 5.5858$
neutron mass	$m_p = 939.56563\,\mathrm{MeV}$
g-factor of the neutron	$g = -3.8261$
deuteron mass	$m_d = 1875.61339\,\mathrm{MeV}$

4.1.1.5 Astronomical constants

earth mass	$m_{\text{♁}} = 5.9736 \times 10^{24}\,\text{kg}$
earth radius	$R_{\text{♁}} = 6370\,\text{km}$
earth gravity	$g_{\text{♁}} = 9.80665\,\text{m/s}$
lunar mass	$m_{☾} = 7.348 \times 10^{22}\,\text{kg}$
lunar radius	$R_{☾} = 1740\,\text{km}$
lunar gravity	$g_{☾} = 1.62\,\text{m/s}$
distance earth-moon	$d_{ES} = 384000\,\text{km}$
sun massa	$m_{☼} = 1.99 \times 10^{30}\,\text{kg}$
sun radius	$R_{☼} = 695300\,\text{km}$
sun gravity	$g_{☼} = 273\,\text{m/s}$
distance earth-sun	$d_{ES} = 1.496 \times 10^{8}\,\text{km}$
sinodic day	$d_{syn} = 24\,\text{h}$
sideric day	$d_{syn} = 23.9345\,\text{h} = 23\,\text{h}\,56\,\text{min}\,4\,\text{s}$
sinodic month	$mon_{syn} = 29.530590\,\text{d}$
sideric month	$mon_{sid} = 27.321666\,\text{d}$
sideric year	$a_{syn} = 365.256365\,\text{h} = 365\,\text{d}\,6\,\text{h}\,9\,\text{min}\,10\,\text{s}$
lunar day	$d_{lunar} = 24.8412\,\text{h}$
$\frac{1}{mon_{sid}}$	$= \frac{1}{a_{sid}} + \frac{1}{mon_{syn}}$
$\frac{1}{d_{sid}}$	$= \frac{1}{a_{sid}} + \frac{1}{d_{syn}}$
$\frac{1}{d_{sid}}$	$= \frac{1}{mon_{sid}} + \frac{1}{d_{lunar}}$

4.1.2 Units

charge	Q	basic unit
current	I	A=C/s
voltage	U	V=N/As
polarizability	α_{pol}	Asm^2/V
susceptibility	χ	1
dipolar moment	1 Debye	$= 10^{-27}/2.998\ \text{Cm} = 10^{-19}/c\ \text{Cm}^2/\text{s} = 39.36\ ea_B$

4.2 Quantities and formulas in classical mechanics

time	t	basic unit
position	$\mathbf{r}$	basic unit
velocity	$\mathbf{v}$	$\mathbf{v} = \dot{\mathbf{r}}$
acceleration	$\mathbf{a}$	$\mathbf{a} = \dot{\mathbf{v}}$
mass	m	basic unit
linear momentum	$\mathbf{p}$	$\mathbf{p} = m\mathbf{v}$
force	$\mathbf{F}$	$\mathbf{F} = \dot{\mathbf{p}} = m\mathbf{a}$
kinetic energy	E_{kin}	$E_{kin} = \frac{m}{2}v^2$
angle	$\vec{\phi}$	basic unit
angular velocity	$\vec{\omega}$	$\vec{\omega} = \dot{\mathbf{r}}$
angular acceleration	$\vec{\alpha}$	$\vec{\alpha} = \dot{\vec{\omega}}$
inertial moment (continuous density)	I	$I = \int r_\perp^2 dm = \int_V \rho(\mathbf{r})[\mathbf{r}^2 - (\mathbf{r} \cdot \hat{\mathbf{e}}_\omega)]dV$
inertial moment (discrete density)	I	$I = \sum_i m_i r_i^2$
angular momentum	$\mathbf{L}$	$\mathbf{L} = I\vec{\omega} = \mathbf{r} \times \mathbf{p}$
torque	$\vec{\tau}$	$\vec{\tau} = \dot{\mathbf{L}} = I\vec{\alpha} = \mathbf{r} \times \mathbf{F}$
rotational energy	E_{rot}	$E_{rot} = \frac{m}{2}\omega^2 r^2$
potential energy	E_{pot}	$E_{pot}^{grav} = mgh$, $E_{pot}^{sprng} = \frac{k}{2}x^2$
work	W	$W = \int_{s_1}^{s_2} \mathbf{F} \cdot d\mathbf{s}$
power	P	$P = \dot{W}$

4.2.1 Particular forces

gravitation	$F_{grav} = mg$
Hooke's for elastic spring	$F_{mola} = -k\Delta x$
friction	$F_{at} = -\mu N$
Stokes' friction	$F_{fr} = -\gamma v$
Newton's friction	$F_{fr} = -\gamma v^2$

4.2.2 Inertial momentum

Steiner's theorem	$I_{\omega_2} = I_{\omega_1} + md^2$, where d is the distance between parallel axes
theorem of perpendicular axes	$I_z = I_x + I_y$ para $\rho(\mathbf{r}) = \delta(z)\sigma(x,y)$

4.2.3 Inertial forces due to transitions to translated and rotated systems

transformation to an accelerated frame	$\mathbf{F}_{Gal} = -m\mathbf{a}$
centrifugal force	$\mathbf{F}_{cf} = -m\vec{\omega} \times (\vec{\omega} \times \mathbf{r})$
Coriolis force	$\mathbf{F}_{Cor} = -2m\vec{\omega} \times \mathbf{v}$

4.2.4 Conservation laws

energy conservation	$\sum_k E_{kin}^{(ini)} + \sum_k E_{pot}^{(ini)} = \sum_k E_{kin}^{(fin)} + \sum_k E_{pot}^{(fin)}$
linear momentum conservation	$\sum_k \mathbf{p}_k^{(ini)} = \sum_k \mathbf{p}_k^{(fin)}$
angular momentum conservation	$\sum_k \mathbf{L}_k^{(ini)} = \sum_k \mathbf{L}_k^{(fin)}$
definition of the center-of-mass	$\mathbf{r}_{cm} \equiv \frac{\sum_k m_k \mathbf{r}_k}{\sum_k m_k}$

4.2.5 Rigid bodies, minimum required number of equations of motion

1. estimate number of moving masses	m_1, m_2, ...
2. identify possible movement (degree of freedom) for every mass	$v_{1x}, v_{2x}, ...$ $v_{1y}, v_{2y}, ...$ $v_{1z}, v_{2z}, ...$ $\omega_1, \omega_2, ...$
3. write down for every degree of freedom an equation of motion	$m\dot{v}_{kl} = \sum_j F_j$ $I\dot{\omega}_k = \sum_j \tau_j$

4.2.6 Gravitational laws

Newton's law	$\mathbf{F}(\mathbf{r}) = -\frac{GMm}{\lvert\mathbf{R}-\mathbf{r}\rvert^2}\hat{\mathbf{e}}_{Rr} = -\nabla V(\mathbf{r})$
gravitational potential	$V(\mathbf{r}) = -\int \frac{Gm}{\lvert\mathbf{r}-\mathbf{r}'\rvert^2}\rho(\mathbf{r}')dV'$

4.2.7 Volume elements

cartesian coordinates	$dV = dxdydz$
cylindrical coordinates	$dV = \rho d\rho d\phi dz$
spherical coordinates	$dV = r^2 \sin\theta dr d\theta d\phi$

4.2.8 Oscillations $ma + bv + kx = F_0 \cos\omega t$

dissipative motion	$k = 0,\ F_0 = 0$	$x(t) = Ae^{-\gamma t},\ \gamma = \frac{b}{2m}$
harmonic oscillation	$b = 0,\ F_0 = 0$	$x(t) = A\cos(\omega_0 t + \delta),\ \omega_0 = \sqrt{\frac{k}{m}}$
damped oscillation	$F_0 = 0$	$x(t) = Ae^{-\gamma t}\cos(\omega t + \delta),\ \omega = \sqrt{\omega_0^2 - \gamma^2}$
forced oscillation		$x(t) = A\cos(\omega t + \delta)\ ,\ A = \frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2+b^2\omega^2}}\ ,$ $\tan\delta = \frac{b\omega}{m(\omega_0^2-\omega^2)}$

4.3 Probability distributions

The *binomial distribution* is defined by,

$$B_k^{(n)} = \binom{n}{k} p^k (1-p)^{n-k}\ . \tag{4.1}$$

The *Poisson distribution* is defined by,

$$P_k = \frac{\lambda^k}{k!} e^{-\lambda}\ , \tag{4.2}$$

for large n and small p, we get $B_k^{(n)} \simeq P_k$ with $\lambda = np$.

4.3.1 Some useful formulae

If the limits of two functions tend to 0, $\lim_{t\to t_0} f(t) = 0 = \lim_{t\to t_0} g(t)$ a rule called *l'Hôpital's rule* goes like,

$$\lim_{t\to t_0} \frac{f(t)}{g(t)} = \lim_{t\to t_0} \frac{f'(t)}{g'(t)}\ . \tag{4.3}$$

Bibliography

[1] A. E. Siegman, *Lasers*, 1986, ISBN.

[2] J. Weiner and P.-T. Ho, *Light-matter interaction, fundamentals and applications*, John Wiley & Sons, Hokoken, New Jersey, 2003, ISBN.

Index